MINIMAL NUMBER OF GENERATORS
OF SOME CLASSES OF GROUPS

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ABSTRACT

The rank of a group G, denoted by $d(G)$, is defined. It is the cardinal of any minimal set of generators of G if G is finitely generated, otherwise G is said to have infinite rank.

Let G be a group with the presentation

$$G = \mathrm{gp}\left(a_1, a_2, \ldots, a_t;\ a_1^{p_1} = a_2^{p_2} = \ldots = a_t^{p_t} = a_1 a_2 \ldots a_t = 1\right),$$

where each of the p_is occurring in the set of relations is a prime. It is shown that any permutation of the generators in the last relator $a_1 a_2 \ldots a_t$ does not change the group. Let there be s different primes occurring in the presentation. It is shown that there is no loss of generality in assuming the first s primes, namely, $p_1, p_2, \ldots, p_s$, to be all distinct from one another.

Let $n = p_1 p_2 \ldots p_s$ and $r_1, r_2, \ldots, r_s$ be the number of times each p_i, $i = 1, 2, \ldots, s$, occurs in the presentation. Again, it is shown that there is no loss of generality in assuming $r_1 \le r_2 \le \ldots \le r_s$.

If now G^n is the group generated by all elements of G, each raised to the nth power, then with the above assumptions, it is proved that

(i) if $s = 1$, then $d\left(G/G^n\right) = t - 1$;

(ii) if $s \ge 2$ and $r_i \ge 2$ for some $1 \le i < s$, then

$$d\left(G/G^n\right) \ge r_s + 1 - \frac{2}{p_1 p_2 \cdots p_{s-1}};$$

(iii) if $s \geq 2$ and $r_i = 1$ for all $1 \leq i < s$, but $r_s \geq 2$, then

$$d\left(G/G^n\right) \geq \max\{1, r_s-2\} + 1 - \frac{2}{p_s p_{s-2}},$$

where $p_0 = 1$.

Let $A \text{ wr } B$ denote the standard restricted wreath product of the abelian groups A and B, called the wreath product of A and B for short. The rank of $A \text{ wr } B$ is established. When the ranks of both A and B are finite, it is equal to the quantity

$$\max_{\text{all } p,\alpha,\beta} \{d_p(A\alpha)+1, d_p(A\alpha)+d_p(B\beta)\},$$

where the maximum is evaluated over all primes p, while α ranges over all homomorphisms of A, and β ranges over all homomorphisms of B. Here $d_p(X)$, for $X = A\alpha, B\beta$, is the p-rank or the minimal number of generators of the Sylow p-subgroup of the abelian group X. If, either A or B or both, have infinite rank, then $A \text{ wr } B$ has infinite rank also.

Also determined is the rank of $A \text{ wr } B$, for the case where A is a finitely generated abelian group and B is the dihedral group of order 2^t, $t \geq 2$.

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CHAPTER 1

INTRODUCTION

A group is either finitely generated or infinitely generated. A set of generators of a finitely generated group G is minimal if no set with fewer elements generates G. The cardinal of any minimal set of generators of G is the minimal number of generators of G, denoted by $d(G)$. For brevity, we shall call $d(G)$ the rank of G. Those groups which can only be infinitely generated, we say have infinite rank.

The rank of a group, in general, is very difficult to determine. However that of an abelian group is well-known. Also, the rank of a free product, in terms of those of its free factors, has been determined by B.H. Neumann [7], while S.A. Meskin [6] has found the rank of certain quotient groups of a class of groups with a single defining relator.

We attempt, in the second chapter, to determine the ranks of certain quotient groups of a special class of Fuchsian groups. The results are incomplete, however. (For a presentation of Fuchsian groups in terms of generators and defining relations, see, for instance, L. Greenberg [2].)

Wreath products have been studied, for instance, by B.H. Neumann, Hanna Neumann and P.M. Neumann [8], P.M. Neumann [10], and G. Baumslag [1]. The rank of a wreath product, however, does not appear to have been studied. It is the main objective in Chapters 3, 4 and 5 to establish $d(A \text{ wr } B)$ for the case where both A and

B are abelian, where $A \text{ wr } B$ is the standard restricted wreath product of A and B. In this respect we succeeded. We are able to determine it in terms of the invariants of certain standard decompositions of A and B, the result being best possible in that sense.

Finally, in Chapter 6, we extend this result a little further when we replace B by some well-known finite 2-groups.

1. Some Lemmas

The following two lemmas are known. They will be used later, and we include references by which they may be proved.

LEMMA 1.1. *If G is a finitely generated group, and H is a subgroup of G of finite index $[G : H]$ in G, then*

$$d(H) \leq [G : H]\big(d(G)-1\big) + 1 .$$

Proof. Apply Theorem 2.10 in W. Magnus, A. Karrass and D. Solitar [5].

LEMMA 1.2. *If A is abelian and B is any group, and if*

$\alpha : A \twoheadrightarrow A^*$ *is an epimorphism of A onto A^*,*

$\beta : B \twoheadrightarrow B^*$ *is an epimorphism of B onto B^*,*

then α can be extended in a natural way to an epimorphism

$$\alpha^* : A \text{ wr } B \twoheadrightarrow A^* \text{ wr } B ,$$

and β can be extended naturally to an epimorphism

$$\beta^* : A^* \text{ wr } B \twoheadrightarrow A^* \text{ wr } B^* ,$$

so that there is a natural epimorphism $\alpha^*\beta^*$ of $A \text{ wr } B$ onto $A^* \text{ wr } B^*$.

Proof. Use first result 22.11 in Hanna Neumann [9], pp. 45-46, and then Lemma 3.2 in K.W. Gruenberg [3].

2. Notation

$|S|$ = cardinal of the set S .

$|G|$ = order of the group G .

$|g|$ = order of the element g .

$m|n$ means m divides n .

g.c.d.(m, n) = greatest common divisor of m and n .

$H \le G$ means H is a subgroup of G .

$H \triangleleft G$ means H is a normal subgroup of G .

$[G : H]$ = index of the subgroup H in G .

G' = derived group of G .

G^n = group generated by all elements of G each raised to the power n .

G/H = quotient group of G by the normal subgroup H or G modulo its normal subgroup H .

$\mathrm{gp}(g_1, g_2, \ldots, g_n)$ = group generated by $g_1, g_2, \ldots, g_n$.

$\mathrm{gp}(S)$ = group generated by S .

$g^h = h^{-1}gh$.

$g^{1+h+\ldots+k} = gg^h \ldots g^k$.

$[g, h] = g^{-1}h^{-1}gh = g^{-1+h}$.

$d(G)$ = rank or minimal number of generators of G .

$A \text{ wr } B$ = standard restricted wreath product of A and B .

A^B = base group of $A \text{ wr } B$.

C_r = cyclic group of order r .

CHAPTER 2

LOWER BOUNDS FOR THE RANKS OF CERTAIN QUOTIENT GROUPS OF A CLASS OF FUCHSIAN GROUPS

After presenting a special class of Fuchsian groups G in terms of generators and defining relations, we go on to determine $d(G/G')$. Mainly, the chapter is concerned with obtaining lower bounds for $d(G/G^n)$, for special cases of the integer n.

1. Presentation of a Special Class of Fuchsian Groups

Let G be a group with the presentation

$$G = \mathrm{gp}\left(a_1, a_2, \ldots, a_t;\ a_1^{p_1} = a_2^{p_2} = \ldots = a_t^{p_t} = a_1 a_2 \ldots a_t = 1\right),$$

where each of the p_is occurring in the set of relations is a prime. Thus G is a special type of Fuchsian group.

One might first guess that $d(G) = t - 1$. This is indeed the case; this information being given to me verbally by S.A. Meskin.

One might then guess that permuting the generators in the last relator $a_1 a_2 \ldots a_t$ may change the group G. This, however, is not the case.

THEOREM 2.1. *Suppose* G_1 *and* G_2 *are groups with the presentations*

$$G_1 = \mathrm{gp}\left(a_1, a_2, \ldots, a_t;\ a_1^{p_1} = a_2^{p_2} = \ldots = a_t^{p_t} = a_1 a_2 \ldots a_t = 1\right)$$

and

$$G_2 = \mathrm{gp}\left(a_1, a_2, \ldots, a_t;\ a_1^{p_1} = a_2^{p_2} = \ldots = a_t^{p_t} = a_1^* a_2^* \ldots a_t^* = 1\right),$$

where $a_1^*a_2^* \ldots a_t^*$ *is any permutation of* $a_1a_2 \ldots a_t$. *Then the two presentations define the same group.*

Proof. Let us first consider the case where

$$a_i^* = a_{i+1} \quad \text{and} \quad a_{i+1}^* = a_i \quad \text{for} \quad 1 \le i \le t \ ,$$

and $a_k^* = a_k$ for all $k \ne i, i+1$, $1 \le k \le t$. In other words, the last relator in each of the group presentations differ only in that the positions of a_i and a_{i+1} are interchanged, for a fixed i .

Now rewrite

$$a_1a_2 \ldots a_ia_{i+1} \ldots a_t$$

as

$$a_1a_2 \ldots a_{i-1}a_{i+1}a_{i+1}^{-1}a_ia_{i+1} \ldots a_t \ ,$$

that is

$$a_1a_2 \ldots a_{i-1}a_{i+1}a_i^{a_{i+1}}a_{i+2} \ldots a_t \ .$$

Let $b = a_i^{a_{i+1}}$. Then $a_i = a_{i+1}ba_{i+1}^{-1}$, and $a_i^{p_i} = 1$ if and only if $1 = a_{i+1}b^{p_i}a_{i+1}$, that is, if and only if $b^{p_i} = 1$.

Therefore by a finite sequence of elementary Tietze transformations, (see W. Magnus, A. Karras and D. Solitar [5], pp. 48-53), we can show that

$$G_1 = \mathrm{gp}\Big\{a_1, a_2, \ldots, a_{i-1}, b, a_{i+1}, a_{i+2}, \ldots, a_t;$$

$$a_1^{p_1} = a_2^{p_2} = \ldots = a_{i-1}^{p_{i-1}} = b^{p_i} = a_{i+1}^{p_{i+1}} = a_{i+2}^{p_{i+2}} = \ldots = a_t^{p_t}$$

$$= a_1 a_2 \ldots a_{i-1} a_{i+1} b a_{i+2} \ldots a_t = 1\Big\} .$$

Then by replacing the symbol b by a_i we have

$$G_1 = \mathrm{gp}\Big\{a_1, a_2, \ldots, a_{i-1}, a_i, a_{i+1}, a_{i+2}, \ldots, a_t;$$

$$a_1^{p_1} = a_2^{p_2} = \ldots = a_{i-1}^{p_{i-1}} = a_i^{p_i} = a_{i+1}^{p_{i+1}} = a_{i+2}^{p_{i+2}} = \ldots = a_t^{p_t}$$

$$= a_1 a_2 \ldots a_{i-1} a_{i+1} a_i a_{i+2} \ldots a_t = 1\Big\} ,$$

which is a presentation for G_2 . Therefore G_1 is isomorphic to G_2 .

Next consider the case where

$$a_i^* = a_j \ , \quad a_j^* = a_i \quad \text{for both } i, j \text{ fixed,}$$

$i \neq j$, $1 \leq i, j \leq t$, and

$$a_k^* = a_k \quad \text{for all } k \neq i, j \ , \quad 1 \leq k \leq t \ .$$

Clearly by repeated applications of the above process we can, by a cyclic permutation of the generators between and including the two flanking generators a_i and a_j , bring a_i to the position originally occupied by a_j without changing the group. By the same procedure, this time keeping a_i fixed in its present position, we can bring a_j via the opposite direction, to the position originally occupied by a_i , without again changing the group. Thus we have shown that G_1 is isomorphic to G_2 for the second case.

Finally for the general case where $a_1^* a_2^* \ldots a_t^*$ is any permutation of $a_1 a_2 \ldots a_t$, we interchange the generators in pairs, where at least one member of the pair, after the interchange, is brought to its new position. Evidently after at most a finite number of such pair interchanges, all the generators will be brought to their new positions; and all the time the group remains unaffected. This completes the proof.

2. Rank after Abelianization

Here we determine the cycle structure of G/G', and then deduce $d(G/G')$ from it.

THEOREM 2.2. *Let*

$$G = \mathrm{gp}\left\{a_1, a_2, \ldots, a_t;\ a_1^{p_1} = a_2^{p_2} = \ldots = a_t^{p_t} = a_1 a_2 \ldots a_t = 1\right\}.$$

Suppose the primes $p_1, p_2, \ldots, p_t$ *fall into* k *sets of identical primes. If* $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ *are all the distinct primes and* $r_1, r_2, \ldots, r_k$, *the corresponding cardinals of the sets containing* $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ *respectively, then* G/G' *is the direct product of* $r_1 - 1, r_2 - 1, \ldots, r_k - 1$ *cycles of orders* $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ *respectively.*

Proof. The presentation for G/G' is that of G with the additional defining relations $[a_i, a_j] = 1$ for all $1 \le i, j \le t$. Therefore we may assume

$$p_1 = p_2 = \ldots = p_{i_1},$$

$$p_{i_1+1} = p_{i_1+2} = \ldots = p_{i_2} , \ldots ,$$

$$p_{i_{k-1}+1} = \ldots = p_{i_k} = p_t .$$

Since $a_1a_2 \ldots a_{i_1}a_{i_1+1} \ldots a_t = 1$, therefore

$$a_1a_2 \ldots a_{i_1} = \left(a_{i_1+1} \ldots a_t\right)^{-1} .$$

The order of $a_1a_2 \ldots a_{i_1}$ therefore divides p_{i_1} and $p_{i_2}p_{i_3} \ldots p_{i_k}$, and hence since $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ are all distinct,

$$a_1a_2 \ldots a_{i_1} = 1 ,$$

and hence also

$$a_{i_1+1} \ldots a_t = 1 .$$

By Tietze transformation we replace the relator $a_1a_2 \ldots a_t$ by the relators $a_1a_2 \ldots a_{i_1}$ and $a_{i_1+1} \ldots a_t$. By continuing in this manner, we show that the relator $a_1a_2 \ldots a_t$ can be replaced by the relators

$$a_1a_2 \ldots a_{i_1} , a_{i_1+1} \ldots a_{i_2} , \ldots , a_{i_{k-1}+1} \ldots a_{i_k} .$$

Thus, for instance, the subgroup of G/G' generated by $a_1, a_2, \ldots, a_{i_1}$ has the presentation

$$\mathrm{gp}\left(a_1, a_2, \ldots, a_{i_1} ; a_1^{p_{i_1}} = a_2^{p_{i_1}} = \ldots = a_{i_1}^{p_{i_1}} = a_1a_2 \ldots a_{i_1} = 1\right) ;$$

and this defines a direct product of $r_1 - 1$ cycles of order p_{i_1}. Hence quite clearly G/G' is isomorphic to a direct product of $r_1-1, r_2-1, \ldots, r_k-1$ cycles of orders $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ respectively.

COROLLARY. *Let*

$$G = \mathrm{gp}\left(a_1, a_2, \ldots, a_t ; a_1^{p_1} = a_2^{p_2} = \ldots = a_t^{p_t} = a_1a_2 \ldots a_t = 1\right).$$

Then $d(G/G') = r - 1$, *where* r *is the greatest cardinal of the sets obtained by grouping identical primes that occurred in the relators of* G.

Proof. We use the notation of the preceding theorem. We may assume, without loss of generality, that

$$r_1-1 \le r_2-1 \le \ldots \le r_k-1 .$$

Now observe that G/G' is isomorphic to a direct product of

$$r_1-1,\ (r_2-1)-(r_1-1),\ (r_3-1)-(r_2-1),\ \ldots,\ (r_k-1)-(r_{k-1}-1)$$

cycles of orders

$$p_{i_1}p_{i_2} \ldots p_{i_k},\ p_{i_2}p_{i_3} \ldots p_{i_k},\ p_{i_3} \ldots p_{i_k},\ \ldots,\ p_{i_k}$$

respectively. Thus

$$d(G/G') = (r_1-1) + (r_2-1) - (r_1-1) + (r_3-1) - (r_2-1)$$
$$+ \ldots + (r_k-1) - (r_{k-1}-1) = r_k - 1 .$$

3. Ranks for some Special Cases

In the remainder of the chapter we shall try to obtain lower bounds for $d\left(G/G^n\right)$, $n \ge 2$, where n is the product of all the distinct primes occurring in the given presentation of G, each appearing once and only once in the product.

Let us, for illustration, first consider the group

$$G = \mathrm{gp}\left(a_1,\ a_2,\ a_3;\ a_1^7 = a_2^3 = a_3^3 = a_1a_2a_3 = 1\right),$$

that is

$$G = \mathrm{gp}\left(a_1,\ a_2;\ a_1^7 = a_2^3 = (a_1a_2)^3 = 1\right).$$

Here $n = 7.3 = 21$.

Let

$$G^* = \mathrm{gp}\left(a_1,\ a_2;\ a_1^7 = a_2^3 = 1,\ a_1^{-1}a_2a_1 = a_1a_2\right).$$

Then G maps homomorphically onto G^* . Moreover G/G^{21} maps homomorphically onto $G^*/(G^*)^{21}$, whence $d\left(G/G^{21}\right) \geq d\left(G^*/(G^*)^{21}\right)$.

In G^* , since $a_1^{-1}a_2a_1 = a_1a_2$, we have the following sequence of implications:

$$a_2a_1a_2^{-1} = a_1^2\ ,\quad a_2a_1^3a_2^{-1} = a_1^6 = \left(a_1^{-1}\right),$$

$$a_2a_1^{-3}a_2^{-1} = a_1\ ,\quad a_1^4 = a_2^{-1}a_1a_2\ .$$

Therefore G^* is an extension of a 7-cycle by a 3-cycle; and its order is $7.3 = 21$, so that $(G^*)^{21}$ is the trivial group.

Since 4 is not congruent to 1 modulo 7 , a_1 does not commute with a_2 ; this means G^* is non-abelian, and so $d(G^*) = 2$.

Therefore

$$2 \geq d\left(G/G^{21}\right) \geq d\left(G^*/(G^*)^{21}\right) = d(G^*) = 2\ ,$$

so that $d\left(G/G^{21}\right) = 2$.

THEOREM 2.3. *Let*

$$G = \mathrm{gp}\left\{a_1, a_2, a_3;\ a_1^2 = a_2^2 = a_3^p = a_1a_2a_3 = 1\right\},$$

where $p \neq 2$ *(here* p *need not be a prime). Then* G *is the dihedral group of order* $2p$ *, so that* $d\left(G/G^{2p}\right) = d(G) = 2$.

Proof. The dihedral group of order $2p$ can be presented on two generators s, t with defining relations

$$s^p = 1\ ,\quad t^2 = 1\ ,\quad t^{-1}st = s^{-1}\ .$$

Now G can be presented on two generators a_1 and a_2 with defining relations

$$a_1^2 = 1\ ,\quad a_2^2 = 1\ ,\quad \left(a_1a_2\right)^p = 1\ .$$

The mapping defined by

$$a_1 \mapsto t\ ,\quad a_2 \mapsto t^{-1}s$$

is a homomorphism from G onto the dihedral group of order $2p$; it is moreover, invertible and is therefore an isomorphism.

Thus G is of order $2p$; hence $G^{2p} = 1$. Therefore $d\left(G/G^{2p}\right) = d(G) = 2$.

4. Reidemeister-Schreier Rewriting Process

We explain, in the next lemma, the principle of a method of writing down a presentation of a special normal subgroup of G , when at least two of the primes occurring are equal.

First we introduce a notation. Let $S(p, s, t)$ denote the class of groups with presentation similar to that of G , where for some $i \neq j$, $1 \leq i, j \leq t$, $p_i = p_j = p, s$, the number of distinct p_is , and t the number of generators appearing in the presentation.

LEMMA 2.4. *If G is in the class $S(p, s, t)$, then G has a normal subgroup H of index p ; moreover H is in $S(q_1, s_1, t_1)$ where q_1 is one of the original p_is , $s_1 \leq s$ and $t_1 = p(t-2)$.*

Proof. On account of Theorem 2.1, there is no loss of generality in taking $p_1 = p_2 = p$. Denote the cyclic group of order p by

$$C_p = \mathrm{gp}(c;\ c^p = 1) .$$

We take H to be the kernel of the homomorphism of G onto C_p defined by

$$a_1 \mapsto c\ ,\quad a_2 \mapsto c^{-1}\ ,\quad a_i \mapsto 1 \text{ for } i = 3, \ldots, t\ .$$

A presentation for H is then obtained by using the Reidemeister-Schreier rewriting process. Such a routine procedure is described in W. Magnus, A. Karrass and D. Solitar [5], pp. 86-95.

Let a_1^j , $j = 0, 1, \ldots, p-1$ be the Schreier representatives.

Generators for H are:

a_1^p with generating symbol x_1 ,

$a_1^j a_2 a_1^{1-j}$ with generating symbols y_j , $j = 0, 1, \ldots, p-1$,

$a_1^j a_i a_1^{-j}$ with generating symbol $z_i(j)$, $i = 3, \ldots, t$,
$j = 0, \ldots, p-1$.

On these generators H is defined by the following relations:

$$x_1 = 1\ ,$$

$$y_0 y_{p-1} y_{p-2} \dots y_3 y_2 y_1 = 1 \ ,$$

$$z_i(j)^{p_i} = 1 \ \text{ for } \ i = 3, \dots, t \ , \ j = 0, 1, \dots, p-1 \ ,$$

$$y_1 z_3(0) z_4(0) \dots z_t(0) = 1 \ ,$$

$$y_2 z_3(1) z_4(1) \dots z_t(1) = 1 \ ,$$

$$y_3 z_3(2) z_4(2) \dots z_t(2) = 1 \ ,$$

$$\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots$$

$$y_{p-1} z_3(p-2) z_4(p-2) \dots z_t(p-2) = 1 \ ,$$

$$y_0 z_3(p-1) z_4(p-1) \dots z_t(p-1) = 1 \ .$$

We now eliminate the generator x_1 and the generator y_j from the presentation to leave the following relations:

$$z_i(j)^{p_i} = 1 \ \text{ for } \ i = 3, \dots, t \ , \ j = 0, 1, \dots, p-1 \ ,$$

$$\begin{aligned} &z_3(0) z_4(0) \dots z_t(0) z_3(1) z_4(1) \dots z_t(1) z_3(2) z_4(2) \dots \\ &\qquad \dots z_t(2) \dots z_3(p-2) z_4(p-2) \dots z_t(p-2) z_3(p-1) z_4(p-1) \dots \\ &\qquad\qquad \dots z_t(p-1) = 1 \ . \end{aligned}$$

Therefore H is in $S\big(p_j, s_1, p(t-2)\big)$ for each $j = 3, \dots, t$.

5. Main Theorem

We make repeated applications of Lemma 2.4 to get a sequence of quotient groups, and then to determine a lower bound for $d\big(G/G^n\big)$.

THEOREM 2.5. *Let*

$$G = \text{gp}\Big\{a_1, a_2, \dots, a_t ; \ a_1^{p_1} = a_2^{p_2} = \dots = a_t^{p_t} = a_1 a_2 \dots a_t = 1\Big\} \ ,$$

where the $p_i s$ *are primes. Suppose there are* s *different primes occurring in the presentation.*

On account of Theorem 2.1 *and by using Tietze transformations there is no loss of generality in assuming the first* s *primes, namely,* $p_1, p_2, \ldots, p_s$, *to be all distinct from one another.*

Let $n = p_1 p_2 \ldots p_s$ *and* $r_1, r_2, \ldots, r_s$ *be the number of times each* p_i , $i = 1, 2, \ldots, s$, *occurs in the presentation.*

On account of Theorem 2.1 *and using Tietze transformations again, there is no loss of generality in assuming* $r_1 \le r_2 \le \ldots \le r_s$.

(i) *If* $s = 1$, *then* $d(G/G^n) = t - 1$.

(ii) *If* $s \ge 2$ *and* $r_i \ge 2$ *for some* $1 \le i < s$, *then*

$$d(G/G^n) \ge r_s + 1 - \frac{2}{p_1 p_2 \cdots p_{s-1}} .$$

(iii) *If* $s \ge 2$ *and* $r_i = 1$ *for all* $1 \le i < s$, *but* $r_s \ge 2$; *then*

$$d(G/G^n) \ge \max\{1, r_s - 2\} + 1 - \frac{2}{p_s \cdot p_{s-2}} ,$$

with the proviso that $p_0 = 1$.

Proof. In terms of the generators $a_1, a_2, \ldots, a_t$, the group G/G^n is defined by the relations

$$a_1^{p_1} = a_2^{p_2} = \ldots = a_t^{p_t} = a_1 a_2 \ldots a_t = x^n = 1 ,$$

where x is any word on the $a_i s$.

The proof of (i) is easy.

To prove (ii) and (iii), first define

$$n_1 = (p_1^*)^{-1}n \ , \quad n_2 = (p_2^*)^{-1}n_1, \ \ldots, \ n_{s-1} = (p_{s-1}^*)^{-1}n_{s-2} \ ,$$

where $p_1^*p_2^* \ldots p_{s-1}^*$ is any permutation of $p_1p_2 \ldots p_{s-1}$. Roughly speaking, n_1 has one prime "missing", n_2 has two primes "missing", and so on.

Proof of (i). If now we take $p_1^* = p_i$, keeping i fixed, then repeated applications of Lemma 2.4 yield a normal series of groups

$$G \triangleright H_1 \triangleright H_2 \triangleright \ldots \triangleright H_{s-2} \triangleright H_{s-1} \ ,$$

with indices

$$[G : H_1] = p_1^* \ , \quad [H_1 : H_2] = p_2^*, \ \ldots, \ [H_{s-2} : H_{s-1}] = p_{s-1}^* \ .$$

Also,

$$G^n \leq H_1^{n_1} \leq H_2^{n_2} \leq \ldots \leq H_{s-2}^{n_{s-2}} \leq H_{s-1}^{n_{s-1}} \ .$$

Consider the sequence of quotient groups

$$G/G^n, \ H_1/H_1^{n_1}, \ H_2/H_2^{n_2}, \ \ldots, \ H_{s-2}/H_{s-2}^{n_{s-2}}, \ H_{s-1}/H_{s-1}^{n_{s-1}} \ .$$

In particular, consider $H_{s-1}/H_{s-1}^{n_{s-1}}$. It is in $S(p_s, 1, r_sp_1^*p_2^* \ldots p_{s-1}^*)$. Again,

$$d\left(H_{s-1}/H_{s-1}^{n_{s-1}}\right) = d\left(H_{s-1}/H_{s-1}^{n_{s-1}}.H'_{s-1}\right)$$

$$= r_sp_1^*p_2^* \ldots p_{s-1}^*-1 = r_sp_1p_2 \ldots p_{s-1}-1 \ ,$$

since $p_1^*p_2^* \ldots p_{s-1}^*$ is a permutation of $p_1p_2 \ldots p_{s-1}$.

Since H_{s-1}/G^n maps homomorphically onto $H_{s-1}/H_{s-1}^{n_{s-1}}$,

$d\left(H_{s-1}/G^n\right) \geq r_s p_1 p_2 \ldots p_{s-1} - 1$, and since

$$\left[G/G^n : H_{s-1}/G^n\right] = [G : H_1][H_1 : H_2] \ldots [H_{s-2} : H_{s-1}]$$
$$= p_1^* p_2^* \ldots p_{s-1}^*$$
$$= p_1 p_2 \ldots p_{s-1} ,$$

applying Lemma 1.1 yields the inequality

$$d\left(H_{s-1}/G^n\right) \leq p_1 p_2 \ldots p_{s-1}\left(d\left(G/G^n\right)-1\right) + 1 .$$

Therefore

$$r_s p_1 p_2 \ldots p_{s-1} - 1 \leq p_1 p_2 \ldots p_{s-1}\left(d\left(G/G^n\right)-1\right) + 1 ,$$

or

$$r_s - \frac{2}{p_1 p_2 \cdots p_{s-1}} + 1 \leq d\left(G/G^n\right) .$$

Proof of (iii). In this case $s \geq 2$, $r_i = 1$ for all $1 \leq i < s$, but $r_s \geq 2$.

First we put a further restriction on r_s , namely that $r_s - 2 < 2$. Then as in proof of (ii), we can get subgroups H_1, H_2 and H_3 (from applications of Lemma 2.4), where

$$G \rhd H_1 \rhd H_2 \rhd H_3 ,$$

and

$$[G : H_1] = p_s , \quad [H_1 : H_2] = p_{s-2} , \quad [H_2 : H_3] = p_{s-1} ,$$

with the proviso that $p_0 = 1$. This condition $p_0 = 1$ occurs when s is exactly equal to 2 , and is taken to mean $H_1 = H_2$.

Hence we get

$$d(G/G^n) \geq r_{s-1} + 1 - \frac{2}{p_s p_{s-2}} .$$

Now $r_{s-1} = 1$, and $r_s - 2 = 0$ or 1; so $\max\{1, r_s-2\} = r_{s-1}$.

Therefore we may replace the inequality by

$$d(G/G^n) \geq \max\{1, r_s-2\} + 1 - \frac{2}{p_s p_{s-2}} .$$

Finally we restrict r_s so that $r_s - 2 \geq 2$. Here we consider $G/G^{p_s}G'$. This has $d\left(G/G^{p_s}G'\right) = r_s - 1$. Since $G^n \leq G^{p_s} \leq G^{p_s}G'$, G/G^n maps homomorphically onto $G/G^{p_s}G'$. Therefore $d(G/G^n) \geq r_s - 1$; or to put it differently

$$d(G/G^n) \geq \max\{1, r_s-2\} + 1 ,$$

since $\max\{1, r_s-2\} = r_s - 2$. Hence, *a fortiori*,

$$d(G/G^n) \geq \max\{1, r_s-2\} + 1 - \frac{2}{p_s p_{s-2}} ,$$

with the proviso that $p_0 = 1$.

Combining the two different sub-cases, we may sum as follows: If $s \geq 2$ and $r_i = 1$ for all $1 \leq i < s$, but $r_s \geq 2$, then

$$d(G/G^n) \geq \max\{1, r_s-2\} + 1 - \frac{2}{p_s p_{s-2}} ,$$

with the proviso that $p_0 = 1$. This completes the proof of the theorem.

6. Some Applications

As an illustration of Theorem 2.5, consider the group

$$G = \mathrm{gp}\left(a_1, a_2, a_3, a_4, a_5; a_1^3 = a_2^3 = a_3^5 = a_4^5 = a_5^5 = a_1a_2a_3a_4a_5 = 1\right).$$

Here $s = 2$, $r_1 = 2$, and $n = 3.5 = 15$. By formula (ii) of the theorem, $d\left(G/G^n\right) \geq 3 + 1 - \frac{2}{3} = 3\frac{1}{3}$, that is $d\left(G/G^n\right) \geq 4$, and so $d\left(G/G^n\right) = 4$.

For a second illustration, take

$$G = \mathrm{gp}\left(a_1, a_2, a_3, a_4, a_5, a_6; a_1^5 = a_2^2 = a_3^2 = a_4^3 = a_5^3 = a_6^3 = a_1a_2a_3a_4a_5a_6 = 1\right).$$

Here $s = 2$, $r_1 = 1$, $r_2 = 2$, $r_3 = 3$, and $n = 5.2.3 = 30$. Again by formula (ii), we have

$$d\left(G/G^n\right) \geq 3 + 1 - \frac{2}{2.5} = 3\tfrac{4}{5} ,$$

that is $d\left(G/G^n\right) \geq 4$.

For another illustration, consider

$$G = \mathrm{gp}\left(a_1, a_2, a_3; a_1^{11} = a_2^7 = a_3^7 = a_1a_2a_3\right) = 1 .$$

In this example, $n = 11.7 = 77$ and $s = 2$. By formula (iii), since $p_s = 7$, $p_{s-2} = 1$, $r_s = 2$, and $\max\{1, r_s-2\} = 1$, we get

$$d\left(G/G^n\right) \geq 1 + 1 - \tfrac{2}{7} = 1\tfrac{5}{7} .$$

Therefore $d\left(G/G^n\right) \geq 2$, and indeed $d\left(G/G^n\right) = 2$.

Yet another illustration, we consider

$$G = \mathrm{gp}\left(a_1, a_2, a_3, a_4, a_5, a_6; a_1^5 = a_2^{17} = a_3^2 = a_4^2 = a_5^2 = a_6^2 = a_1a_2a_3a_4a_5a_6 = 1\right).$$

Here $n = 5.17.2 = 170$, $s = 3$, $r_1 = r_2 = 1$, and $r_3 = 4$. By formula (iii), taking $p_3 = 2$ and p_1 (that is p_{s-2} in this case) $= 5$, we get that

$$\begin{aligned} d\left(G/G^n\right) &\geq \max\{1, r_3-2\} + 1 - \frac{2}{p_3p_1} \\ &= \max\{1, 2\} + 1 - \frac{2}{2.5} \\ &= 3 - \tfrac{1}{5} = 2\tfrac{4}{5} , \end{aligned}$$

so $d\left(G/G^n\right) \geq 3$.

CHAPTER 3

LOWER BOUNDS FOR THE RANKS OF WREATH PRODUCTS OF ABELIAN GROUPS

The wreath product of two groups is briefly mentioned, some results of a preliminary nature are proved, and a lower bound for the rank of the wreath product of any two finitely generated abelian groups is established.

1. Wreath Product of Groups

The standard restricted wreath product of the groups A and B, denote $A \text{ wr } B$ and hereafter called wreath product of A and B for short, may be viewed as the set of all pairs bf for $b \in B$, $f \in A^B$, where A^B is the group of all functions from B to A under componentwise multiplication, where multiplication of pairs is defined by

$$bf.cg = (bc)\left(f^c g\right) ,$$

for $b, c \in B$ and $f, g \in A^B$, and where f^c is the function from B to A given by

$$f^c(y) = f\left(yc^{-1}\right)$$

for all $y \in B$.

DEFINITION. Let B be generated by a set of elements $\{b_i \mid i \in I\}$, where I is some index set and the b_is are all elements of B. Denote $K = \text{gp}\left(\{b_i g_i \mid i \in I\}\right)$, where the g_is are all elements of A^B.

The group K together with A^B generates $A \text{ wr } B$.

LEMMA 3.1. *The group* $A \text{ wr } B$ *is* KA^B .

Proof. Since $A \text{ wr } B = BA^B$, the result follows.

Further, when A is abelian, $K \cap A^B$ is a normal subgroup.

LEMMA 3.2. *If* A *is abelian, then* $K \cap A^B$ *is normal in* $A \text{ wr } B$.

Proof. Let $g \in K \cap A^B$ and $w \in A \text{ wr } B$. Write $w = kf$, where $k \in K$ and $f \in A^B$, from Lemma 3.1. Then

$$\begin{aligned} w^{-1}gw &= (kf)^{-1}g(kf) \\ &= f^{-1}k^{-1}gkf \\ &= f^{-1}g_1f , \end{aligned}$$

where $g_1 = k^{-1}gk \in K \cap A^B$, since $g \in K \cap A^B$ and A^B is normal. Again A^B is abelian, so

$$f^{-1}g_1f = g_1 .$$

Therefore $w^{-1}gw \in K \cap A^B$, and the result follows.

THEOREM 3.3. *If* A *and* B *are any two abelian groups, and if at least one of* A *and* B *has infinite rank, then* $A \text{ wr } B$ *has also infinite rank.*

Proof. If A has infinite rank, we map B homomorphically onto the trivial group E . Then apply Lemma 1.2 to get epimorphism

$$\alpha : A \text{ wr } B \twoheadrightarrow A \text{ wr } E = A .$$

If B has infinite rank, we map A homomorphically onto E , and again apply Lemma 1.2 to get epimorphism

$$\beta : A \text{ wr } B \twoheadrightarrow E \text{ wr } B = B .$$

2. Standard Decomposition of Abelian Groups

If G is a finitely generated abelian group, then G can be decomposed in a standard way into a direct product of cycles (see A.G. Kurosh [4], pp. 145-151). In such a decomposition, the cycles are so arranged that those of finite orders occur at the beginning, in the order of increasing magnitude so that the order of the first cycle divides that of the second, that of the second divides the third and so on, and those cycles of infinite orders, henceforth regarded as zero orders, occur at the end.

Let now A and B be any two finitely generated abelian groups. With respect to such a standard decomposition, let

$$A = C_{r_1} \times C_{r_2} \times \ldots \times C_{r_{m_1}} \times \ldots \times C_{r_{m_1+m_2}} \times \ldots \times C_{r_{m_1+m_2+m_3}} , \quad (1)$$

where

$$r_1|r_2| \ldots |r_{m_1}| \ldots |r_{m_1+m_2} ,$$

and

$$B = C_{s_1} \times C_{s_2} \times \ldots \times C_{s_{n_1}} \times \ldots \times C_{s_{n_1+n_2}} \times \ldots \times C_{s_{n_1+n_2+n_3}} , \quad (2)$$

where

$$s_1|s_2| \ldots |s_{n_1}| \ldots |s_{n_1+n_2} .$$

With respect to each standard decomposition, the non-zero orders divide upwards in a natural way. However, as between the two decompositions, there is a first finite cycle of non-zero order in each decomposition for which the greatest common divisor of this pair of orders is not 1 , or such a situation might not arise. We shall presently distinguish them. We shall also distinguish the infinite cycles from the finite ones in each of the decompositions.

Let the pairs of non-negative integers

$$\left(m_1, n_1\right) , \left(m_2, n_2\right) , \left(m_3, n_3\right)$$

be such that

$$m_1 + m_2 + m_3 = m = d(A) ,$$

$$n_1 + n_2 + n_3 = n = d(B) ,$$

and for which

$$\text{g.c.d.}\left(r_i, s_j\right) = 1$$

for all $0 < i \le m_1$, $0 < j \le n_1$,

$$r_i \ne 0 , \quad s_j \ne 0 , \quad \text{g.c.d.}\left(r_i, s_j\right) \ne 1$$

for all $m_1 < i \le m_1+m_2$, $n_1 < j \le n_1+n_2$, and

$$r_i = s_j = 0$$

for all $m_1+m_2 < i \le m_1+m_2+m_3$, $n_1+n_2 < j \le n_1+n_2+n_3$.

This formulation indicates that, among the finite cycles, there are m_1 of the first decomposition which have no common divisor with n_1 of the second, all these cycles coming first in their respective decomposition, m_2 cycles of the first which have common divisors with n_2 of the second, coming next; and coming at the end, m_3 and n_3 infinite cycles of the first and second decomposition respectively.

3. Lower Bounds for $d(A \text{ wr } B)$

The rank of A wr B for two finitely generated abelian groups A and B equals the sum of the ranks of A and B , when there is some common prime dividing the orders of each and every cycle in their standard decompositions.

THEOREM 3.4. *Let* A *and* B *be any two finitely generated abelian groups with standard decompositions* (1) *and* (2) *respectively. If* $\text{g.c.d.}(r_1, s_1) \neq 1$ *, then*

$$d(A \text{ wr } B) = d(A) + d(B) .$$

Proof. Write $t = \text{g.c.d.}(r_1, s_1)$. Consider

$$A^* = \text{gp}\left(a_1, a_2, \ldots, a_m; a_i^t = [a_i, a_j] = 1, 1 \leq i, j \leq m\right) ,$$

$$B^* = \text{gp}\left(b_1, b_2, \ldots, b_n; b_i^t = [b_i, b_j] = 1, 1 \leq i, j \leq n\right) .$$

Apply Lemma 1.2 to map $A \text{ wr } B$ homomorphically onto $A^* \text{ wr } B^*$ to get

$$d(A) + d(B) \geq d(A \text{ wr } B) \geq d(A^* \text{ wr } B^*) .$$

But

$$A^* \text{ wr } B^* = \text{gp}\left(A, B; \left[a_i, a_j^y\right] = 1, 1 \leq i, j \leq m, y \in B\right) ,$$

therefore

$$(A^* \text{ wr } B^*)/(A^* \text{ wr } B^*)'$$

$$= \text{gp}\left(A^* \text{ wr } B^*; [a_i, b_j] = 1, 1 \leq i \leq m, 1 \leq j \leq n\right) ,$$

which is a direct product of $d(A) + d(B)$ cycles, each of order t . Therefore $d(A \text{ wr } B) = d(A) + d(B)$.

The next lemma gives a lower bound for $d(A \text{ wr } B)$ in terms of m_1, m_2, m_3, n_1, n_2 and n_3 .

LEMMA 3.5. *If* A *and* B *are any two finitely generated abelian groups with standard decompositions* (1) *and* (2), *then*

$$d(A \text{ wr } B) \geq \max\{m_1+m_2+m_3+1, m_1+m_2+m_3+n_3, m_3+n_1+n_2+n_3, m_2+m_3+n_2+n_3\} .$$

Proof. Map A homomorphically onto A_1, the direct product of m copies of p-cycles, where p is any fixed prime dividing r_1. Again map B homomorphically onto B_1 the direct product of n copies of q-cycles, where q is any fixed prime dividing s_1.

The base group $A_1^{B_1}$ of A_1 wr B_1 consists of $|B_1|$ copies of A_1. Thus $d\left(A_1^{B_1}\right) = m|B_1|$. Also, $\left[A_1 \text{ wr } B_1 : A_1^{B_1}\right] = |B_1|$. Apply now Lemma 1.1 to get

$$d\left(A_1^{B_1}\right) \le \left[A_1 \text{ wr } B_1 : A_1^{B_1}\right]\left(d\left(A_1 \text{ wr } B_1\right)-1\right) + 1 ,$$

that is

$$m|B_1| \le |B_1|\left(d\left(A_1 \text{ wr } B_1\right)-1\right) + 1$$

that is

$$m - \frac{1}{|B_1|} + 1 \le d\left(A_1 \text{ wr } B_1\right) .$$

Since B_1 is not trivial, this means that

$$d\left(A_1 \text{ wr } B_1\right) \ge m + 1 .$$

Therefore from Lemma 1.2 we get that

$$d(A \text{ wr } B) \ge d\left(A_1 \text{ wr } B_1\right) \ge m + 1 .$$

Now map B homomorphically onto B_1 the direct product of n_3 r_1-cycles and then apply Lemma 1.2 and Theorem 3.4 to get

$$d(A \text{ wr } B) \ge d\left(A \text{ wr } B_1\right) = m + n_3 .$$

By a symmetrical argument we get also

$$d(A \text{ wr } B) \ge d\left(A_1 \text{ wr } B\right) = m_3 + n ,$$

where A_1 is the direct product of m_3 s_1-cycles.

Finally by mapping A homomorphically onto A_2 the direct product of $m_2 + m_3$ copies of g.c.d.$\left(r_{m_1+1}, s_{n_1+1}\right)$-cycles, and B homomorphically onto B_2 the direct product of $n_2 + n_3$ copies of g.c.d.$\left(r_{m_1+1}, s_{n_1+1}\right)$-cycles, and applying Lemma 1.2 and Theorem 3.4 once more, we have

$$d(A \text{ wr } B) \geq d\left(A_2 \text{ wr } B_2\right) = m_2 + m_3 + n_2 + n_3 .$$

Combining all four cases we get the result.

CHAPTER 4

RANKS OF SOME WREATH PRODUCTS OF ABELIAN GROUPS OF CO-PRIME ORDERS

From Lemma 3.5, it would appear that for some special cases of the finitely generated abelian groups A and B,

$d(A \text{ wr } B) = \max\{m_1+m_2+m_3+1,\ m_1+m_2+m_3+n_3,\ m_3+n_1+n_2+n_3,\ m_2+m_3+n_2+n_3\}$.

We shall provide evidence in support of the statement.

1. Definition

Throughout this chapter let

$$A = C_r = \mathrm{gp}\left(a;\ a^r = 1\right) ,$$

$$B = C_s^2 = \mathrm{gp}\left(b,\ c;\ b^s = c^s = [b,\ c] = 1\right) ,$$

$$W = A \text{ wr } B .$$

Let also f be the function of the base group A^B for which

$$f(1) = a ,$$

and

$$f(y) = 1 \quad \text{for all} \quad y \in B ,\quad y \neq 1 .$$

2. Ranks of C_r wr C_s^2 for $s = 2, 3, 5$ and where g.c.d.$(r, s) = 1$

For a fixed s, $d\left(C_r \text{ wr } C_s^2\right)$, g.c.d.$(r, s) = 1$, is determined in turn for $s = 2, 3$ and 5. They are found to agree with the formula stated at the beginning of the chapter.

THEOREM 4.1. *The rank of* C_r wr C_2^2, g.c.d.$(r, 2) = 1$, *is* 2 .

Proof. Put $K = \mathrm{gp}(bf, cf^b)$. By Lemma 3.1, $W = KA^B$. We shall show that $K \cap A^B = A^B$, so that $W = K$, whence from Lemma 3.5, $d(W) = 2$.

The elements $[bf, cf^b]$ and $(bf)^2$ are in $K \cap A^B$, where

$$[bf, cf^b] = f^{-2+b+c} ,$$

$$(bf)^2 = f^{1+b} .$$

The product of the first and the inverse of the second element is

$$f^{-3+c} , \tag{i}$$

again in $K \cap A^B$. Since by Lemma 3.2, $K \cap A^B \lhd W$, we get on conjugating f^{-3+c} by c , that

$$f^{-3c+1} \tag{ii}$$

is in $K \cap A^B$.

From (i) and (ii) we get

$$f^{-9+3c} . f^{-3c+1} = f^{-8}$$

is in $K \cap A^B$. Since g.c.d.$(|f|, 2) = 1$, this implies $f \in K \cap A^B$, and from the normality of $K \cap A^B$, that f^b, f^c and f^{bc} are also in $K \cap A^B$. Hence $A^B = K \cap A^B$.

THEOREM 4.2. *The rank of* C_r wr C_3^2 , g.c.d.$(r, 3) = 1$, *is* 2 .

Proof. The argument proceeds on similar lines as in Theorem 4.1. We let $K = \mathrm{gp}(bf, cf^b)$, and consider

$$[bf, cf^b] = f^{-1-b^2+c+b} ,$$

$$(bf)^3 = f^{1+b+b^2} .$$

The product of these two elements is f^{2b+c} .

Now conjugate f^{2b+c} in turn by c^2, $-2bc$ and $4b^2$ to get respectively

$$f^{2bc^2+1} , \quad f^{-4b^2c-2bc^2} , \quad f^{8+4b^2c} .$$

Take their product, we have

$$f^{2bc^2+1} . f^{-4b^2c-2bc^2} . f^{8+4b^2c} = f^9 .$$

This element f^9 is in $K \cap A^B$. Since g.c.d.$(|f|, 3) = 1$, it follows that $f \in K \cap A^B$, and from the normality of $K \cap A^B$, conclude that $A^B = K \cap A^B$. Thus K generates W .

THEOREM 4.3. *The rank of* $C_r \text{ wr } C_5^2$, g.c.d.$(r, 5) = 1$, *is* 2 .

Proof. Again put $K = \text{gp}\left(bf, cf^b\right)$. Consider

$$F_1 = \left[bf, cf^b\right] = f^{-1-b^2+c+b} = f^{c-1+b-b^2} ,$$

$$F_2 = (bf)^5 = f^{1+b+b^2+b^3+b^4} ,$$

$$F_3 = \left[bf, cf^b\right]^{c^4} = f^{1-c^4+bc^4-b^2c^4} .$$

The exponent sum of f in F_3 is a polynomial in b and c whose coefficients are integers modulo r . Now the degree of c in this polynomial is 4 . What is needed is to form the product of F_3 with suitable multiple of conjugates of F_1 to get an f , whose exponent sum is again a polynomial in b and c , and having integers modulo r as coefficients, but this time with the degree of

c equal or less than 3 . This is achieved as follows:

$$F_4 = F_3 F_1^{c^3} F_1^{-bc^3} F_1^{b^2c^3} ,$$

and on simplification,

$$F_4 = f^{1-c^3+2bc^3-3b^2c^3+2b^3c^3-b^4c^3} .$$

Ultimately we want to get an f whose exponent sum is a polynomial only of b . Evidently by proceeding along the above lines, we have in succession

$$F_5 = F_4 F_1^{c^2} F_1^{-2bc^2} F_1^{3b^2c^2} F_1^{-2b^3c^2} F_1^{b^4c^2}$$

$$= f^{1+2c^2+2bc^2-6b^2c^2+7b^3c^2-6b^4c^2} ,$$

$$F_6 = F_5 F_1^{-2c} F_1^{-2bc} F_1^{6b^2c} F_1^{-7b^3c} F_1^{6b^4c}$$

$$= f^{1+15c-6bc-6b^2c+15b^3c-19b^4c} ,$$

and

$$F_7 = F_6 F_1^{-15} F_1^{6b} F_1^{6b^2} F_1^{-15b^3} F_1^{19b^4}$$

$$= f^{1+49-40b+15b^2+15b^3-40b^4}$$

$$= f^{50-40b+15b^2+15b^3-40b^4} .$$

The next step is to force out f^1 or some multiple of one of its conjugates by powers of b , by forming products from multiples of conjugates of F_7 by powers of b as many times as required, and where necessary also including F_2 into the products. The following steps lead to our goal.

$$F_8 = F_7F_7^b = f^{10+10b-25b^2+30b^3-25b^4} ,$$

$$F_9 = F_8F_8^b = f^{-15+20b-15b^2+5b^3+5b^4} ,$$

$$F_{10} = F_9F_9^b = f^{-10+5b+5b^2-10b^3+10b^4} ,$$

$$F_{11} = F_{10}F_{10}^b = f^{-5b+10b^2-5b^3} ,$$

$$F_{12} = F_{11}F_{11}^b = f^{-5b+5b^2+5b^3-5b^4} ,$$

$$F_{13} = F_{12}F_{12}^b = f^{-5b+10b^3-5} .$$

Therefore

$$F_{13}F_{11}^bF_2^5 = f^{-5b+10b^3-5}f^{-5b^2+10b^3-5b^4}f^{5+5b+5b^2+5b^3+5b^4}$$

$$= f^{25b^3} .$$

So far all the steps show that $f^{25b^3} \in K \cap A^B$. Since g.c.d.($|f|$, 5) = 1 , this implies that $f^{b^3} \in K \cap A^B$, and from the normality of $K \cap A^B$, conclude that $A^B = K \cap A^B$.

3. Some Lemmas

It might perhaps be claimed that in general for g.c.d.(r, s) = 1 , C_r wr C_s^2 is always generated by bf and cf^b . If one tries to prove this for C_r wr C_6^2 , g.c.d.(r, 6) = 1 or for C_r wr C_7^2 , g.c.d.(r, 7) = 1 , by arguing along the same lines as in Theorem 4.3, then the degree of complexity is greater, and in fact I do not know whether or not this argument would work. However for $r = 5$ and $s = 4$, the two elements bf and cf^b do not generate C_5 wr C_4^2 . We shall show this after first proving two lemmas.

LEMMA 4.4. *Let* $W = C_r \operatorname{wr} C_s^2$. *Put* $A = C_r$ *and* $B = C_s^2$, *and let* $\{b, c\}$ *generate* C_s^2 . *Let further* $K = \operatorname{gp}(bg, ch)$, *where* g *and* h *are any two functions of* A^B . *Then*

$$K \cap A^B = \operatorname{gp}\left((bg)^s, (ch)^s, [bg, ch]^x \mid x \in \operatorname{gp}(b, c)\right) .$$

Proof. Since $K \cap A^B$ is normal in W , quite clearly the group on the right hand side is contained in $K \cap A^B$.

Suppose $y \in K \cap A^B$. Then since $y \in K$, for some positive integer t ,

$$y = \prod_{i=1}^{t} (bg)^{\beta_i} (ch)^{\gamma_i} .$$

Since y also belongs to A^B , and moreover since B is abelian, we have that

$$\sum_{i=1}^{t} \beta_i = st_1 \quad \text{and} \quad \sum_{i=1}^{t} \gamma_i = st_2 ,$$

for some integers t_1 and t_2 .

In the expression for y , we shall group, at the front, all the bg terms $(st_1$ in all$)$, followed in the middle by the ch terms $(st_2$ in all$)$, and leaving at the back an expression consisting solely of conjugates of $[bg, ch]$ by elements of B . This is achieved as follows:

Suppose the factors $(ch)(bg)$ is present somewhere in the expression for y , say, for instance

$$y = y_1(ch)(bg)y_2 ,$$

for some expressions y_1 and y_2 of powers of ch and bg. Then

$$y = y_1(bg)(ch)[ch, bg]y_2 .$$

But y_2 is expressible as

$$y_2 = y_2^* y_2^{**} ,$$

where

$$y_2^* \in B \text{ and } y_2^{**} \in A^B .$$

Since $[ch, bg] \in K \cap A^B \triangleleft W$ and A^B is abelian, therefore

$$[ch, bg]^{y_2^* y_2^{**}} = [ch, bg]^{y_2^*}$$

and hence that

$$[ch, bg]y_2^* y_2^{**} = y_2^* y_2^{**}[ch, bg]^{y_2^*} ,$$

that is,

$$[ch, bg]y_2 = y_2[ch, bg]^{y_2^*} .$$

This means that

$$y = y_1(bg)(ch)y_2[ch, bg]^{y_2^*} .$$

Clearly by repeated applications, and after at most a finite number of steps, we would achieve what is desired. This will then show that y belongs to the group on the right hand side. Thus $K \cap A^B \leq$ r.h.s.

LEMMA 4.5. *Let* $W = C_r \text{ wr } C_s^2$, g.c.d.$(r, s) = 1$. *As before put* $A = C_r$, $B = C_s^2$ *and let* $\{b, c\}$ *generate* C_s^2. *Let further* $K = \text{gp}\left(bf, cf^b\right)$, *where* f *is the function of* A^B *for which*

$$f(1) = a \text{ where } a \text{ generates } C_r ,$$

$$f(y) = 1 \text{ for all } y \in B , \ y \neq 1 .$$

Suppose τ *is any one solution of the congruence equation*

$$s\tau \equiv 1 \bmod r .$$

Let now

$$b^* = 1 + b + b^2 + \ldots + b^{s-1} .$$

Then $K \cap A^B$ *is the group*

$$K \cap A^B = \mathrm{gp}\left\{\left([bf, cf^b]f^{\tau b^*}\right)^x \mid x \in \mathrm{gp}(b, c)\right\} .$$

Proof. Put

$$K_1 = \mathrm{gp}\left\{\left([bf, cf^b]\right)^x \mid x \in \mathrm{gp}(b, c)\right\} ,$$

$$K_2 = \mathrm{p}\left\{\left([bf, cf^b]f^{\tau b^*}\right)^x \mid x \in \mathrm{gp}(b, c)\right\} .$$

Now

$$(bf)^s = f^{1+b+b^2+\ldots+b^{s-1}} = f^{b^*} ,$$

$$[bf, cf^b] = f^{-1-b^2+c+b} = f_1 .$$

Both elements belong to $K \cap A^B$, and therefore

$$f_2 = f^{-1-b^2+c+b}f^{\tau b^*}$$

also belongs to $K \cap A^B$. By the normality of $K \cap A^B$, also all the conjugates of f_2 by elements of B are in $K \cap A^B$. Therefore

$$K_2 \leq K \cap A^B .$$

Taking the product of all the conjugates of f_2 by $1, b, b^2, \ldots, b^{s-1}$, we get

$$f^{-b^*+b^*c+s\tau b^*} = f^{b^*c+(s\tau-1)b^*}$$

$$= f^{b^*c} ,$$

since $s\tau \equiv 1 \bmod r$.

Therefore taking the product of all the conjugates of f_2 by the appropriate sequence of elements of B , we get in turn

$$f^{b^*c}, f^{b^*c^2}, f^{b^*c^3}, \ldots, f^{b^*c^{s-1}}, f^{b^*} .$$

The product of all these elements is

$$f^{b^*c^*} ,$$

where

$$c^* = 1 + c + c^2 + \ldots + c^{s-1} .$$

Each of these elements belongs to K_2 , and therefore

$$f^{-\tau b^*c}, f^{-\tau b^*c^2}, f^{-\tau b^*c^3}, \ldots, f^{-\tau b^*c^{s-1}}, f^{-\tau b^*}$$

all belong to K_2 .

By multiplying each conjugate of f_2 in K_2 by the appropriate element from the preceding sequence of elements, we may get the corresponding conjugate of f_1 in K_1 . (For instance for f_2^{bc} in K_2 , we choose $f^{-\tau b^*c}$, since

$$f_2^{bc}f^{-\tau b^*c} = f^{(-1-b^2+c+b)bc}f^{\tau b^*bc}f^{-\tau b^*c}$$

$$= f_1^{bc} ,$$

as $f^{\tau b^*bc} = f^{\tau b^*c}$.)

This means that $K_1 \le K_2$.

Now consider

$$f_1 f_1^c f_1^{c^2} \cdots f_1^{c^{s-1}} .$$

This element is

$$f^{(-1-b^2+c+b)c^*} = f^{-c^*+c^*+(b-b^2)c^*}$$
$$= f^{(b-b^2)c^*} .$$

By reasoning symmetrically we get also

$$f^{(b^2-b^3)c^*},\ f^{(b^3-b^4)c^*},\ \ldots,\ f^{(b^{s-2}-b^{s-1})c^*},\ f^{(b^{s-1}-1)c^*},\ f^{(1-b)c^*} .$$

From $f^{(b-b^2)c^*}$ and $f^{(b^2-b^3)c^*}$ we get

$$f^{(b-b^3)c^*} .$$

Thus by iteration we can get the sequence

$$f^{(b-b^2)c^*},\ f^{(b-b^3)c^*},\ \ldots,\ f^{(b-b^{s-1})c^*},\ f^{(b-1)c^*} .$$

The product of all these elements is

$$f^{(sb-b^*)c^*} .$$

This element is in K_1 and hence in K_2. But we have already seen that $f^{b^*c^*} \in K_2$, therefore $f^{sbc^*} \in K_2$, and since g.c.d.$(r, s) = 1$, this implies $f^{bc^*} \in K_2$. But $f^{bc^*} = (cf^b)^s$. Also since f^{b^*} (that is $(bf)^s$) is in K_2, and $K_1 \le K_2$, we now apply Lemma 4.4 to conclude that $K \cap A^B = K_2$. This completes the proof.

We apply Lemma 4.5 to show that bf and cf^b do not generate $C_5 \text{ wr } C_4^2$. Put $r = 5$, $s = 4$, $\tau = -1$. We enumerate the generators of K_2 (16 in all) in the following table below which has the element

$$([bf,\ cf^{b}]f^{-b*})^{b^{i-1}c^{j-1}}$$

in the $(i,\ j)$th entry, where $1 \le i \le 4$ and $1 \le j \le 4$. We write down for each entry only the exponent sum of f.

$-2-2b^2-b^3+c$, $-2c-2b^2c-b^3c+c^2$, $-2c^2-2b^2c^2-b^3c^2+c^3$, $-2c^3-2b^2c^3-b^3c^3+1$

$-2b-2b^3-1+bc$, $-2bc-2b^3c-c+bc^2$, $-2bc^2-2b^3c^2-c^2+bc^3$, $-2bc^3-2b^3c^3-c^3+b$

$-2b^2-2-b+b^2c$, $-2b^2c-2c-bc+b^2c^2$, $-2b^2c^2-2c^2-bc^2+b^2c^3$, $-2b^2c^3-2c^3-bc^3+b^2$

$-2b^3-2b-b^2+b^3c$, $-2b^3c-2bc-b^2c+b^3c^2$, $-2b^3c^2-2bc^2-b^2c^2+b^3c^3$, $-2b^3c^3-2bc^3-b^2c^3+b^3$

Now $K \cap A^B$ has a set of independent generators K^* consisting of all the conjugates of f by elements of B. We consider a titled exponent sum matrix. This is a matrix of integers with its rows labelled by this set of independent generators K^*, its columns labelled by the set of generators K_2 which are words of K^*, and such that the $(i,\ j)$th entry in the matrix is the exponent sum of the jth word (column label) relative to the ith generator (row label). (See Appendix in which this matrix appears first.)

We use the method of W. Magnus, A. Karras and D. Solitar [5], pp. 140-144, to diagonalize the matrix. After diagonalization, the resultant matrix is one in which along the principal diagonal, there occur ones on all entries except the (14, 14), (15, 15) and (16, 16)th, where each entry there is zero. All the off principal diagonal entries are zeros. This shows that bf and cf^{b} do not generate $C_5 \text{ wr } C_4^2$. (See Appendix for details of the calculations. For each entry, the addition is carried out modulo 5 .)

By the same method we can also show that bf and cf^{b^3} do not generate $C_5 \text{ wr } C_4^2$; for Lemma 4.5 remains valid if we replace the element cf^{b} in K by cf^{b^3} .

It is interesting, however, to compare this result with the next one.

THEOREM 4.6. *Let* g.c.d.$(r, 10) = 1$, $A = C_r$, $B = C_4^2$ *and* $W = A \text{ wr } B$. *Further let* $K = \left\{bf, cf^{b^3}\right\}$. *Then* K *generates* W *so that* $d\left(C_r \text{ wr } C_4^2\right) = 2$ *for* g.c.d.$(r, 10) = 1$.

Proof. Consider $\left[bf, cf^{b^3}\right]$ and apply the method of Theorem 4.3 to show $f \in K \cap A^B$. It will then follow that K generates W . We omit the steps.

4. Rank of $C_3^4 \text{ wr } C_2^5$

In this section we determine the rank of $C_3^4 \text{ wr } C_2^5$ by producing a set of generators of a certain form. The method of proof is *ad hoc*, and its inclusion here is merely for the sake of interest.

THEOREM 4.7. *Rank of* $C_3^4 \text{ wr } C_2^5$ *is* 5 .

Proof. Put $A = C_3^4$, $B = C_2^5$. Let f_i be the functions of A^B for which

$$f_i(1) = a_i\ ,\quad i = 1, 2, 3, 4\ ,$$

$$f_i(y) = 1 \quad \text{for all } y \in B\ ,\ y \neq 1\ ,$$

and where a_1, a_2, a_3, a_4 are generators of the cycles of A .

Let b_1, b_2, b_3, b_4, b_5 be generators of the cycles of B .

Finally let

$$K = \mathrm{gp}\Big(b_1f_1, b_2f_2, b_3f_3, b_4f_4,$$

$$b_5f_1^{b_1}f_2^{(1+b_1)2b_2}f_3^{(1+b_1)(1+b_2)b_3}f_4^{(1+b_1)(1+b_2)(1+b_3)2b_4}\Big)$$

$$\left[b_1f_1, b_5f_1^{b_1}f_2^{(1+b_1)2b_2}f_3^{(1+b_1)(1+b_2)b_3}f_4^{(1+b_1)(1+b_2)(1+b_3)2b_4}\right]$$

$$= \left[b_1f_1, b_5f_1^{b_1}\right]$$

$$= f_1^{2+2+b_5+b_1} = f_1^{1+b_1+b_5}\ ,$$

$$(b_1f_1)^2 = f_1^{1+b_1}\ .$$

Take the product of the first element with the inverse of the second to get

$$f_1^{b_5} \in K \cap A^B\ .$$

This implies that $f_1 \in K \cap A^B$, and *a fortiori* $f_1 \in K$, and hence that b_1 belongs to K .

Again $f_1^{-b_1} \in K \cap A^B$ and so

$$b_5 f_2^{(1+b_1)2b_2} f_3^{(1+b_1)(1+b_2)b_3} f_4^{(1+b_1)(1+b_2)(1+b_3)2b_4} \in K .$$

Next we show that

$$b_5 f_2^{b_2} f_3^{(1+b_2)2b_3} f_4^{(1+b_2)(1+b_3)b_4} \in K ,$$

and this is seen to be so by considering

$$b_5 f_2^{(1+b_1)2b_2} f_3^{(1+b_1)(1+b_2)b_3} f_4^{(1+b_1)(1+b_2)(1+b_3)2b_4}$$

$$[b_1, b_2 f_2]^{b_1 b_2} [b_1, b_3 f_3]^{(1+b_2)b_3} [b_1, b_4 f_4]^{(1+b_2)(1+b_3)2b_4} ,$$

which, on simplification, is the element in question since

$$[b_1, b_i f_i] = f_i^{1+2b_1} \quad \text{for } i = 2, 3, 4 ,$$

and where addition of the exponent sum of f is carried out modulo 3 .

Now we consider

$$\left[b_2 f_2 , b_5 f_2^{b_2} f_3^{(1+b_2)2b_3} f_4^{(1+b_2)(1+b_3)b_4}\right] ,$$

which is the same as $\left[b_2 f_2 , b_5 f_2^{b_2}\right]$ and on applying the same argument as before, we may show that $f_2 \in K$ and hence $b_2 \in K$.

Evidently by an inductive process, we may show also that f_3, b_3, f_4, b_4 and finally b_5 all belong to K . Thus $K = C_3^4 \text{ wr } C_2^5$ and the result is proved.

Our proof here depends on the arithmetical properties of 3 and

2 . For the more general case $C_r^m \text{ wr } C_s^n$, g.c.d.$(r, s) = 1$, this method may or may not work. In order to avoid too much dependence on r and s , it is apparent that we need to look for a new set of generators with a different form.

CHAPTER 5

RANKS OF WREATH PRODUCTS OF FINITELY GENERATED ABELIAN GROUPS

The rank of A wr B is determined for the finitely generated abelian groups A and B. The principle of the method, each time, is to find a set of generators whose cardinal assumes the value

$$\max\{m_1+m_2+m_3+1,\ m_1+m_2+m_3+n_3,\ m_3+n_1+n_2+n_3,\ m_2+m_3+n_2+n_3\} .$$

1. Definition

Let

$$d = \max\{m_1+m_2+m_3+1,\ m_1+m_2+m_3+n_3,\ m_3+n_1+n_2+n_3,\ m_2+m_3+n_2+n_3\} .$$

Let the elements a_i , $1 \le i \le m$, form a minimal set of generators of A , and the elements b_i , $1 \le i \le n$, form a minimal set of generators of B . Define f_i , $1 \le i \le m$, to be the functions of A^B for which

$$f_i(1) = a_i ,$$

$$f_i(y) = 1 \text{ for all } y \in B ,\ y \ne 1 .$$

Hereafter throughout this chapter we regard both A and B to be finitely generated abelian groups.

2. Rank of C_r^m wr C_s^n , g.c.d.$(r, s) = 1$

It is convenient to determine the rank of C_r^m wr C_s^{m+1} , g.c.d.$(r, s) = 1$, for a fixed m first, and then deduce that of C_r^m wr C_s^n , g.c.d.$(r, s) = 1$, for any two positive integers m and n .

THEOREM 5.1. *The rank of* $C_r^m \operatorname{wr} C_s^{m+1}$, g.c.d.$(r, s) = 1$, *is* $m + 1$.

Proof. Put $A = C_r^m$, $B = C_s^{m+1}$ and $W = C_r^m \operatorname{wr} C_s^{m+1}$. Let

$$b_i^*(s) = 1 + b_i + b_i^2 + \ldots + b_i^{s-1} ,$$

$$b_i^{**}(s) = b_i^*(s) - 1 + b_i ,$$

for $i = 1, 2, \ldots, m+1$. Let τ be any one solution of

$$sx \equiv 1 \bmod r .$$

Consider the set of elements of W , namely

$$K_1 = \left\{ b_1 f_1^{-b_1^{**}(s)} , b_2 f_2^{-b_2^{**}(s)} , \ldots , b_m f_m^{-b_m^{**}(s)} , \right.$$

$$\left. b_{m+1} f_1 f_2^{\tau^1 b_1^*(s)} f_3^{\tau^2 b_1^*(s) b_2^*(s)} \ldots f_m^{\tau^{m-1} b_1^*(s) b_2^*(s) \ldots b_{m-1}^*(s)} \right\} .$$

This set has $m + 1$ elements, and we shall show that it generates W . It will then follow, on account of Lemma 3.5, that K_1 is a minimal set of generators of W .

Let K be the group generated by the elements of K_1 . By Lemma 3.2, $K \cap A^B$ is normal in W .

$$\left(b_i f_i^{-b_i^{**}(s)} \right)^s = f_i^{-b_i^{**}(s) b_i^*(s)} = f_i^{-s b_i^*(s)} ,$$

for all $1 \le i \le m$. Since g.c.d.$(r, s) = 1$, this implies that $f_i^{-b_i^*(s)}$, and hence $f_i^{b_i^*(s)} \in K \cap A^B$, for all $1 \le i \le m$.

Put

$$B_i = b_{m+1} f_i f_{i+1}^{\tau^1 b_i^*(s)} f_{i+2}^{\tau^2 b_i^*(s) b_{i+1}^*(s)} \cdots f_m^{\tau^{m-i} b_i^*(s) b_{i+1}^*(s) \ldots b_{m-1}^*(s)} ,$$

for $1 \le i \le m$. Then

$$\left[b_1 f_1^{-b_1^{**}(s)}, B_1\right] = \left[b_1 f_1^{-b_1^{**}(s)}, b_{m+1} f_1 f_2^{\tau^1 b_1^*(s)} f_3^{\tau^2 b_1^*(s) b_2^*(s)} \cdots \right.$$

$$\left. \cdots f_m^{\tau^{m-1} b_1^*(s) b_2^*(s) \ldots b_{m-1}^*(s)}\right]$$

$$= \left[b_1 f_1^{-b_1^{**}(s)}, b_{m+1} f_1\right]$$

$$= f_1^{b_1^{**}(s) - b_1 - b_1^{**}(s) b_{m+1} + 1}$$

$$= f_1^{b_1^*(s) - b_1^{**}(s) b_{m+1}} .$$

Since $f_1^{b_1^*(s)} \in K \cap A^B$ and by the normality of $K \cap A^B$, we have that $f_1^{-b_1^{**}(s)} \in K \cap A^B$. Now

$$f_1^{b_1^*(s)} f_1^{-b_1^{**}(s)} = f_1^{1-b_1} \in K \cap A^B .$$

Conjugate $f_1^{1-b_1}$ by b_1 , we get $f_1^{b_1 - b_1^2} \in K \cap A^B$. Therefore $f_1^{1-b_1} f_1^{b_1 - b_1^2} = f_1^{1-b_1^2} \in K \cap A^B$. Hence by iteration we may show that

$$f_1^{1-b_1}, f_1^{1-b_1^2}, f_1^{1-b_1^3}, \ldots, f_1^{1-b_1^{s-1}}$$

all belong to $K \cap A^B$. Therefore

$$\prod_{k=1}^{s-1} f_1^{1-b_1^k} = f_1^{s-b_1^*(s)} .$$

Because $f_1^{b_1^*(s)} \in K \cap A^B$ and g.c.d.$(r, s) = 1$, we get that

$$f_1 \in K \cap A^B .$$

Therefore b_1 belongs to K .

Thus since $f_i^{b_i^*(s)} \in K \cap A^B$ for all $2 \le i \le m$, from the normality of $K \cap A^B$, it follows that

$$f_i^{\left(1-b_1^j\right)b_i^*(s)} \in K \cap A^B , \tag{i}$$

for all $2 \le i \le m$ and $1 \le j \le s-1$. Again

$$\left[b_i f_i^{-b_i^{**}(s)}, b_1^j\right] = f_i^{\left(1-b_1^j\right)b_i^{**}(s)} \in K \cap A^B ,$$

for all $2 \le i \le m$ and $1 \le j \le s-1$. Therefore for all $2 \le i \le m$ and $1 \le j \le s-1$,

$$f_i^{\left(1-b_1^j\right)b_i^*(s)} f_i^{-\left(1-b_1^j\right)b_i^{**}(s)} \in K \cap A^B ,$$

that is,

$$f_i^{\left(1-b_1^j\right)\left(1-b_i\right)} \in K \cap A^B .$$

The conjugate by b_i of this element is

$$f_i^{\left(1-b_1^j\right)\left(b_i-b_i^2\right)} .$$

Therefore

$$f_i^{\left(1-b_1^j\right)\left(1-b_i\right)} f_i^{\left(1-b_1^j\right)\left(b_i-b_i^2\right)} = f_i^{\left(1-b_1^j\right)\left(1-b_i^2\right)} .$$

Evidently by an inductive process we may show that

$$f_i^{\left(1-b_1^j\right)\left(1-b_i\right)} , f_i^{\left(1-b_1^j\right)\left(1-b_i^2\right)} , \ldots , f_i^{\left(1-b_1^j\right)\left(1-b_i^{s-1}\right)}$$

all belong to $K \cap A^B$.

$$\prod_{k=1}^{s-1} f_i^{\left(1-b_1^j\right)\left(1-b_i^k\right)} = f_i^{\left(1-b_1^j\right)\left(s-b_i^*(s)\right)}$$

$$= f_i^{\left(1-b_1^j\right)s} f_i^{-\left(1-b_1^j\right)b_i^*(s)} ,$$

and from (i), we get that

$$f_i^{\left(1-b_1^j\right)s} \in K \cap A^B ,$$

that is,

$$f_i^{\left(1-b_1^j\right)} \in K \cap A^B \quad \text{since} \quad \text{g.c.d.}(r, s) = 1 .$$

Therefore

$$\prod_{j=1}^{s-1} f_i^{\left(1-b_1^j\right)} = f_i^{s-b_1^*(s)} \in K \cap A^B ,$$

for all $2 \le i \le m$.

Thus $f_2^{\tau\left(s-b_1^*(s)\right)} \in K \cap A^B$, and on using the fact that $K \cap A^B$ is normal in W repeatedly, we have also

$$f_i^{\tau^{i-1}\left(s-b_1^*(s)\right)b_2^*(s)b_3^*(s)\ldots b_{i-1}^*(s)} \in K \cap A^B ,$$

for all $3 \le i \le m$.

Then

$$B_1 f_1^{-1} f_2^{\tau(s-b_1^*(s))} \prod_{i=3}^{m} f_i^{\tau^{i-1}(s-b_1^*(s))b_2^*(s)b_3^*(s)\ldots b_{i-1}^*(s)} = B_2 \in K ,$$

since $f_2^{\tau b_1^*(s)} f_2^{\tau(s-b_1^*(s))} = f_2^{\tau s} = f_2'$, as $\tau s \equiv 1 \bmod r$, and for a typical i in $3 \le i \le m$,

$$f_i^{\tau^{i-1}b_1^*(s)b_2^*(s)\ldots b_{i-1}^*(s)} f_i^{\tau^{i-1}(s-b_1^*(s))b_2^*(s)\ldots b_{i-1}^*(s)}$$

$$= f_i^{\tau^{i-2}(\tau b_1^*(s)+\tau s-\tau b_1^*(s))b_2^*(s)\ldots b_{i-1}^*(s)}$$

$$= f_i^{\tau^{i-2}b_2^*(s)\ldots b_{i-1}^*(s)} ,$$

as $\tau s \equiv 1 \bmod r$.

Next we consider the set of elements

$$K_2 = \left\{ b_2 f_2^{-b_2^{**}(s)} , b_3 f_3^{-b_3^{**}(s)} , \ldots , b_m f_m^{-b_m^{**}(s)} , \right.$$

$$\left. b_{m+1} f_2 f_3^{\tau b_2^*(s)} \ldots f_m^{\tau^{m-2}b_2^*(s)\ldots b_{m-1}^*(s)} \right\}$$

together with the commutator

$$\left[b_2 f_2^{-b_2^{**}(s)} , B_2 \right] ,$$

and repeat similarly as before.

It will now be clear how we are to proceed with the proof. We apply the above process repeatedly to show, in succession, that

$$f_1, b_1, f_2, b_2, \ldots, f_m, b_m$$

and lastly b_{m+1} belong to K . This will then complete the proof.

COROLLARY. *The rank of* C_r^m wr C_s^n , g.c.d.$(r, s) = 1$, *is* $\max\{m+1, n\}$.

Proof. Here $d = \max\{m+1, n\}$ since $m_2 = n_2 = m_3 = n_3 = 0$ so that $m_1 = m$ and $n_1 = n$. Suppose m_0 is $\min\{m+1, n\}$. We need merely to consider the critical case $C_r^{m_0-1}$ wr $C_s^{m_0}$ as in the theorem, with the proviso that this time K_1 is modified by the addition of a required number of b_is or f_is to make up the total of $\max\{m+1, n\}$ elements.

3. Rank of A wr B

The standard decompositions of the finitely generated abelian groups A and B , on recapitulation, are respectively

$$A = C_{r_1} \times C_{r_2} \times \ldots \times C_{r_{m_1}} \times \ldots \times C_{r_{m_1+m_2}} \times \ldots \times C_{r_{m_1+m_2+m_3}} , \quad (1)$$

$$B = C_{s_1} \times C_{s_2} \times \ldots \times C_{s_{n_1}} \times \ldots \times C_{s_{n_1+n_2}} \times \ldots \times C_{s_{n_1+n_2+n_3}} , \quad (2)$$

where

$$r_1 | r_2 | \ \ldots \ | r_{m_1} | \ \ldots \ | r_{m_1+m_2} \quad \text{and} \quad s_1 | s_2 | \ \ldots \ | s_{n_1} | \ \ldots \ | s_{n_1+n_2} .$$

Also we have

$$m = m_1 + m_2 + m_3 = d(A) ,$$

$$n = n_1 + n_2 + n_3 = d(B) ,$$

for which

$$\text{g.c.d.}\left(r_i, s_j\right) = 1$$

for all $0 < i \le m_1$, $0 < j \le n_1$;

$$r_i \neq 0 \ , \quad s_j \neq 0 \ , \quad \text{g.c.d.}(r_i, s_j) \neq 1$$

for all $m_1 < i \leq m_1+m_2$, $n_1 < j \leq n_1+n_2$; and

$$r_i = s_j = 0$$

for all $m_1+m_2 < i \leq m_1+m_2+m_3$, $n_1+n_2 < j \leq n_1+n_2+n_3$.

Again for convenience, we shall first determine the rank of A wr B for the case when both A and B have no infinite cycles in their standard decompositions.

THEOREM 5.2. *Let A and B be any two finitely generated abelian groups with standard decompositions* (1) *and* (2) *respectively. Suppose that both A and B are finite. Then the rank of A wr B is*

$$\max\{m_1+m_2+1, \ n_1+n_2, \ m_2+n_2\} \ .$$

Proof. Since both A and B are finite, this means that $m_3 = n_3 = 0$ so that $m = m_1 + m_2$ and $n = n_1 + n_2$. In this case d reduces to $\max\{m_1+m_2+1, n_1+n_2, m_2+n_2\}$.

We use, with modifications, the method of Theorem 5.1.

This time let

$$b_i^*(s_i) = 1 + b_i + b_i^2 + \ldots + b_i^{s_i-1} \ ,$$

$$b_i^{**}(s_i) = b_i^*(s_i) - 1 + b_i \ ,$$

for all $1 \leq i \leq n$.

Consider the set of elements

$$\left\{c_1h_1^{-c_1^{**}(s_1)},\ c_2h_2^{-c_2^{**}(s_2)},\ \ldots,\ c_kh_k^{-c_k^{**}(s_k)},\right.$$

$$c_{k+1}h_1h_2^{\tau_1c_1^*(s_1)}h_3^{\tau_1\tau_2c_1^*(s_1)c_2^*(s_2)}\cdots$$

$$\left.\cdots h_k^{\tau_1\tau_2\ldots\tau_{k-1}c_1^*(s_1)c_2^*(s_2)\ldots c_{k-1}^*(s_{k-1})}\right\},$$

where k is an integer to be determined, where each of the c_is stand for a different b_i, each of the h_is stand for a different f_i, where g.c.d.$(|c_i|, |h_j|) = 1$ for all $1 \leq i \leq k$ and all $i \leq j \leq k$, and where τ_i, for all $1 \leq i \leq k-1$, is any one solution of the set of congruence equations

$$|c_i|x \equiv 1 \bmod |h_j|,$$

for all $i \leq j \leq k$.

The integer k is to be chosen in such a way that after the addition to this set of the remaining b_is and f_is that do not appear in it, the new set K_1 so obtained would have cardinal equal to d.

It can be checked that, with certain essential modifications at a number of places, the method of Theorem 5.1 goes through. That is to say, K_1 generates a group K, which group contains all the b_is and f_is.

Our proof, therefore, reduces to that of finding k, identifying the c_is and h_is for all $1 \leq i \leq k$ from among the b_is and f_is respectively, identifying c_{k+1} from the among the b_is and adding in to K_1, the remaining b_is and f_is that have not been so identified.

To do this, we consider several case distinctions.

Case 1: $m_2 \geq n_1$ *and* $m_1 < n_2$.

Here $d = m_2 + n_2$. Choose $k = n_1 + m_1$. Form $c_i h_i^{-c_i^{**}(s_i)}$, $i = 1, 2, \ldots, n_1$, by pairing elements of $\left\{b_1, b_2, \ldots, b_{n_1}\right\}$ with elements of $\left\{f_{m_1+1}, f_{m_1+2}, \ldots, f_{m_1+n_1}\right\}$. Form $c_i h_i^{-c_i^{**}(s_i)}$, $i = n_1+1, n_1+2, \ldots, n_1+m_1$, by pairing elements of $\left\{b_{n_1+1}, b_{n_1+2}, \ldots, b_{n_1+m_1}\right\}$ with elements of $\left\{f_1, f_2, \ldots, f_{m_1}\right\}$.

Choose $c_{k+1} = b_{n_1+m_1+1}$. Add to K_1 the b_is and f_is not already taken on. There are $n_2 - m_1 - 1$ and $m_2 - n_1$ of them respectively. This makes

$$|K_1| = n_1 + m_1 + 1 + n_2 - m_1 - 1 + m_2 - n_1 = m_2 + n_2 .$$

Case 2: $m_2 \geq n_1$ *and* $m_1 \geq n_2$.

Here $d = m_1 + m_2 + 1$ and we may assume $n_2 \neq 0$; for if not, $m_2 = 0$ and therefore $n_1 \leq 0$, yielding $n \leq 0$, an impossibility.

Choose $k = n_1 + n_2 - 1$. Form $c_i h_i^{-c_i^{**}(s_i)}$ for $i = 1, 2, \ldots, n_1$, by pairing elements of $\left\{b_1, b_2, \ldots, b_{n_1}\right\}$ with elements of $\left\{f_{m_1+1}, f_{m_1+2}, \ldots, f_{m_1+n_1}\right\}$. Form $c_i h_i^{-c_i^{**}(s_i)}$ for $i = n_1+1, n_1+2, \ldots, n_1+n_2-1$, by pairing elements of

$\left\{b_{n_1+1}, b_{n_1+2}, \ldots, b_{n_1+n_2-1}\right\}$ with elements of

$\left\{f_1, f_2, \ldots, f_{n_2-1}\right\}$. (This second set of pairs is empty when

$n_2 = 1$.)

Choose $c_{k+1} = b_{n_1+n_2}$. Add to K_1 those f_is not already chosen, of which there are $m_2 - n_1 + m_1 - (n_2-1)$ in all. This gives

$$|K_1| = n_1 + n_2 + m_2 - n_1 + m_1 - n_2 + 1 = m_2 + m_1 + 1 .$$

Case 3: $m_2 < n_1$ *and* $m_1 \leq n_2$.

Here $d = n_1 + n_2$. Choose $k = m_2 + m_1$. We take $c_{k+1} = b_{n_1-1}$ and pair off, firstly, elements of $\left\{b_1, b_2, \ldots, b_{m_2}\right\}$ with elements of $\left\{f_{m_1+1}, f_{m_1+2}, \ldots, f_{m_1+m_2}\right\}$ and, secondly, elements of $\left\{b_{n_1+1}, b_{n_1+2}, \ldots, b_{n_1+m_1}\right\}$ with those of $\left\{f_1, f_2, \ldots, f_{m_1}\right\}$ in that order. Lastly add to K_1 the b_is not already taken (there are $n_1 - m_2 - 1 + n_2 - m_1$ of them). This gives

$$|K_1| = m_2 + m_1 + 1 + n_1 - m_2 - 1 + n_2 - m_1 = n_1 + n_2 .$$

Case 4: $m_2 < n_1$ *and* $m_1 > n_2$.

Since $m_2 < n_1$ and $m_1 > n_2$, therefore $m_1+m_2 > n_2+m_2$. Here d further reduces to $\max\{m_1+m_2+1, n_1+n_2\}$. We may therefore subdivide Case 4 according as

(i) $m_1+m_2 < n_1+n_2$ or

(ii) $m_1+m_2 \geq n_1+n_2$,

so that d assumes the corresponding values

$$n_1 + n_2 \text{ or } m_1 + m_2 + 1$$

respectively.

In (i) we choose $k = m_1 + m_2$ and take $c_{k+1} = b_1$. We first pair elements of $\left\{b_2, \ldots, b_{m_2+1}\right\}$ with elements of $\left\{f_{m_1+1}, f_{m_1+2}, \ldots, f_{m_1+m_2}\right\}$ and then form m_1 pairs from any m_1 elements of $\left\{b_{m_2+2}, \ldots, b_{n_1+n_2}\right\}$ with elements of $\left\{f_1, f_2, \ldots, f_{m_1}\right\}$. (Note that

$$\left|\left\{b_{m_2+2}, \ldots, b_{n_1+n_2}\right\}\right| \geq \left|\left\{f_1, f_2, \ldots, f_{m_1}\right\}\right| .)$$

Add to K_1 the remaining b_is which have not been already chosen. This will give $|K_1| = n_1 + n_2$.

In (ii) we choose $k = n_1 + n_2$ and again take $c_{k+1} = b_1$. We pair off, firstly, elements of $\left\{b_2, \ldots, b_{m_2+1}\right\}$ with elements of $\left\{f_{m_1+1}, f_{m_1+2}, \ldots, f_{m_1+m_2}\right\}$ and, secondly, form $n_1 + n_2 - m_2 - 1$ pairs from elements of $\left\{b_{m_2+2}, b_{m_2+3}, \ldots, b_{n_1+n_2}\right\}$ with any $n_1 + n_2 - m_2 - 1$ elements of $\left\{f_1, f_2, \ldots, f_{m_1}\right\}$. (Note that this time,

$$\left|\left\{b_{m_2+2}, b_{m_2+3}, \ldots, b_{n_1+n_2}\right\}\right| \leq \left|\left\{f_1, f_2, \ldots, f_{m_1}\right\}\right| .)$$

Add to K_1 the remaining f_is not already paired. This will give $|K_1| = m_1 + m_2 + 1$.

COROLLARY. *Let* A *and* B *be any two finitely generated abelian groups with standard decompositions* (1) *and* (2) *respectively. Then the rank of* $A \text{ wr } B$ *is*

$$\max\{m_1+m_2+m_3+1,\ m_1+m_2+m_3+n_3,\ m_3+n_1+n_2+n_3,\ m_2+m_3+n_2+n_3\} .$$

Proof. We make the same case distinctions as in Theorem 5.2.

In Cases 1, 3 and subcase (i) of 4, we need only put into K_1 the extra b_is and f_is of zero orders, of which there are n_3 and m_3 of them.

In Case 2 and sub-case (ii) of 4, if $n_3 = 0$ then we put into K_1 the extra f_is (there are m_3 of them) to give $|K_1| = m_1 + m_2 + m_3 + 1$. However, if $n_3 \neq 0$, an essential change is required; but this presents no new difficulty. We merely replace c_{k+1} by any one b_i of zero order, and make the appropriate changes in the pairings, and, after adding to K_1 the extra f_is of zero orders and the remaining extra b_is of zero orders, we would get $|K_1| = m_1 + m_2 + m_3 + n_3$.

CHAPTER 6

RANKS OF SOME OTHER WREATH PRODUCTS

The result of Chapter 5 is extended a little further. We shall also, in conclusion, state a conjecture.

1. Rank of A wr D_{2^t}

Let D_{2^t} , $t \geq 2$, be the dihedral group of order 2^t . In terms of the generators b and c , this group is defined by the relations

$$b^{2^{t-1}} = 1 \ , \quad c^2 = 1 \ , \quad c^{-1}bc = b^{-1} \ .$$

THEOREM 6.1. *Suppose* A *is an* r*-cycle, and* B *the dihedral group of order* 2^t , $t \geq 2$, *as defined above. Then* $d(A \text{ wr } B) = 3$ *or* 2 *according as* g.c.d.$(r, 2) = 2$ *or* 1 .

Proof. If g.c.d.$(r, 2) = 2$, we consider in place of B , the quotient group B^* , where B^* is the group B modulo its derived group B' . Since B^* is abelian, and has the same rank as B , we apply Theorem 3.4 to get that $d(A \text{ wr } B^*) = 3$. Hence $d(A \text{ wr } B) = 3$ also.

If now g.c.d.$(r, 2) = 1$, let f be the function of A^B for which

$$f(1) = a \ , \text{ where } a \text{ is any generator of } A \ ,$$

and

$$f(y) = 1 \text{ for all } y \in B \ , \quad y \neq 1 \ .$$

Let $K = \text{gp}(bf, cf^c)$. By Lemma 3.2 the subgroup $K \cap A^B$ is again normal in A wr B .

Consider the following elements of $K \cap A^B$:

$$[bf,\ cf^c] = [b,\ c]f^{-b^{-1}c^{-1}bc-bc+c+c}$$

$$= b^2 f^{-b^2-bc+2c} \ ,$$

$$(cf^c)^2 = f^{1+c} \ ,$$

$$(cf^c)^{2b^{-1}} = f^{b^{-1}+cb^{-1}} = f^{b^{-1}+bc} \ ,$$

$$(bf)^2 = b^2 f^{1+b} \ .$$

Therefore

$$[bf,\ cf^c](cf^c)^{-4}(cf^c)^{2b^{-1}}(bf)^2 = b^2 f^{-b^2-bc+2c} f^{-2-2c} f^{b^{-1}+bc} b^2 f^{1+b}$$

$$= b^2 f^{-b^2-2+b^{-1}} b^2 f^{1+b}$$

$$= f^{-1-2b^2+b+1+b}$$

$$= f^{-2b^2+2b} \in K \cap A^B \ .$$

Since g.c.d.$(|f|,\ 2) = 1$, this means that $f^{b-b^2} \in K \cap A^B$. Again, $f^{1-b} \in K \cap A^B$, since $K \cap A^B$ is normal. Moreover, $f^{1-b} f^{b-b^2} = f^{1-b^2} \in K \cap A^B$. Hence by iteration we can show, in succession, that

$$f^{1-b^i} \in K \cap A^B \ ,$$

for all $i = 1,\ 2,\ \ldots,\ 2^{t-1}-1$. Therefore

$$\prod_{i=1}^{2^{t-1}-1} f^{1-b^i} = f^{2^{t-1}} f^{-b*} \ ,$$

where $b* = 1 + b + b^2 + \ldots + b^{2^{t-1}-1}$. But $(bf)^{2^{t-1}} = f^{b*} \in K \cap A^B$.

Therefore $f^{2^{t-1}}$ and hence $f \in K \cap A^B$, so that K generates A wr B. This completes the proof.

COROLLARY. *Let* A *be a finitely generated abelian group with standard decomposition* (1). *Let* a_i, $i = 1, 2, \ldots, m$, *be the generators of each of the cycles in the decomposition* (1), *and let* B *be the dihedral group of order* 2^t, $t \geq 2$. *Then*

$$d(A \text{ wr } B) = \max\{d(A)+1, 2\} \text{ or } d(A) + 2 .$$

It is equal to $d(A) + 2$ *if and only if there are no* a_is *for which* $\text{g.c.d.}(|a_i|, 2) = 1$.

Proof. We consider the critical case as in Theorem 6.1, and then fill in the additional generators.

2. Rank of C_r wr Q_8, g.c.d.$(r, 6) = 1$

The quaternion group of order 8 has a presentation

$$Q_8 = \text{gp}\left(b, c;\ b^4 = 1,\ b^2 = c^2,\ c^{-1}bc = b^{-1}\right) .$$

In the next theorem, $d(C_r \text{ wr } Q_8)$ for g.c.d.$(r, 6) = 1$, is obtained. (I am unable to improve the result to the more general case where g.c.d.$(r, 2) = 1$.)

THEOREM 6.2. *Let* A *be an* r*-cycle, and* B *the quaternion group of order* 8. *Then* $d(A \text{ wr } B) = 3$ *or* 2 *according as* g.c.d.$(r, 2) = 2$ *or* g.c.d.$(r, 6) = 1$.

Proof. If g.c.d.$(r, 2) = 2$, then by a similar method as in the previous theorem, we can show that $d(A \text{ wr } B) = 3$.

If g.c.d.$(r, 6) = 1$, let f again be the function of A^B for which

$$f(1) = a \text{ , where } a \text{ is any generator of } A \text{ ,}$$
$$f(y) = 1 \text{ for all } y \in B \text{ , } y \neq 1 \text{ .}$$

Let also $K = \mathrm{gp}\left\{bf^{-(2b+b^2+b^3)},\ cf^{-b^3}\right\}$. By Lemma 3.2, $K \cap A^B$ is normal in $A \text{ wr } B$.

The subsequent expressions involving b and products of conjugates of f are all in K , those consisting only of products of conjugates of f are in $K \cap A^B$. From these we deduce that $f \in K \cap A^B$, whence the result.

$$\left[bf^{-(2b+b^2+b^3)},\ cf^{-b^3}\right]$$
$$= [b,\ c]f^{(2b+b^2+b^3)b^{-1}c^{-1}bc}f^{b^3c^{-1}bc}f^{-(2b+b^2+b^3)c-b^3}$$
$$= [b,\ c]f^{2b^3+1+b+b^2-b^3-(2b+b^2+b^3)c}$$
$$= b^2f^{1+b+b^2+b^3-(2b+b^2+b^3)c} \quad . \tag{i}$$

$$\left[bf^{-(2b+b^2+b^3)}\right]^4 = f^{-4(1+b+b^2+b^3)} \quad .$$

Since g.c.d.$(|f|,\ 2) = 1$, we get

$$f^{1+b+b^2+b^3} \quad . \tag{ii}$$

Since $K \cap A^B$ is normal, we get also

$$f^{(1+b+b^2+b^3)c} \quad . \tag{iii}$$

Therefore from (i), (ii) and (iii), we get

$$b^2f^{(1-b)c} \quad . \tag{iv}$$

Taking its square, we have

$$f^{(b^2-b^3+1-b)c} \quad ,$$

since b^2 commutes with c. Thus we get

$$f^{1-b+b^2-b^3} .$$

Using (ii) and the fact that g.c.d.$(|f|, 2) = 1$, we get also

$$f^{1+b^2} \tag{v}$$

and

$$f^{b+b^3} . \tag{vi}$$

Since b^2 commutes with c,

$$\left[b^2 f^{(1-b)c}, cf^{-b^3}\right] = f^{-(1-b)cb^2c^{-1}b^2c+b^3c^{-1}b^2c+(1-b)c^2-b^3}$$

$$= f^{-(1-b)c+b+b^2-b^3-b^3}$$

$$= f^{-(1-b)c+b+b^2-2b^3} .$$

Form product of (iv) with this element to get

$$b^2 f^{b+b^2-2b^3} . \tag{vii}$$

Now

$$bf^{-\left(2b+b^2+b^3\right)} f^{1+b^2} f^{b+b^3} = bf^{1-b} .$$

Squaring we get $b^2 f^{1-b^2}$, and using (v) we have $b^2 f^2$. Take the product of the inverse of this element with (vii), that is,

$$b^2 f^{-2b^2} b^2 f^{b+b^2-2b^3} = f^{-2+b+b^2-2b^3} .$$

Use (v) and (vi) to get f^{-3+3b}, whence its inverse f^{3-3b}. Since g.c.d.$(|f|, 3) = 1$, we therefore have f^{1-b}.

By an already familiar argument, we also get f^{1-b^2} and

f^{1-b^3} , Therefore

$$f^{1-b}f^{1-b^2}f^{1-b^3} = f^{4-(1+b+b^2+b^3)} ,$$

and using (ii), we have f^4 , and hence f , since g.c.d.$(|f|, 2) = 1$. This completes the proof.

3. Conclusion

In a nilpotent group G , the set of periodic elements forms a normal subgroup of the whole group. Each Sylow p-subgroup of G , p a prime, is unique in G . We can speak of the p-rank of G , denote $d_p(G)$, as the minimal number of generators of its Sylow p-subgroup. Moreover, when G is finite, $d(G)$ is the maximum of all the p-ranks of G , as p ranges over all the primes dividing the group order.

Let α, β be any homomorphisms of the finitely generated abelian groups A and B respectively. Then the result of Chapter 5 can be summarised as

$$d(A \text{ wr } B) = \max_{\text{all } p,\alpha,\beta} \{d_p(A\alpha)+1,\ d_p(A\alpha)+d_p(B\beta)\} .$$

This maximum is to be taken over all primes p , over all homomorphisms α of A and over all homomorphisms β of B .

When B is a finite nilpotent group, not necessarily abelian, B still possesses a minimal set of generators, which elements can be so arranged that their orders divide upwards. Moreover, B has a rank-preserving homomorphism mapping it onto one of its Sylow p-subgroup.

It could be that for a finitely generated abelian group A and a finite nilpotent group B ,

$$d(A \text{ wr } B) = \max_{\text{all } p,\alpha,\beta} \{d_p(A\alpha)+1,\ d_p(A\alpha)+d_p(B\beta)\} ,$$

where the right hand side is evaluated over all primes p , as α and β range over all homomorphisms of A and B respectively. The examples of Theorems 6.1 and 6.2 in the present chapter provide, though little, some support to the conjecture.

APPENDIX

The following matrices show the various stages of diagonalizing the first one.

The addition in each entry is carried out modulo 5 , using the integers 0, ±1, ±2 . Entries left blank denote 0 , so that this symbol appears nowhere in the matrices.

$$\begin{bmatrix}
1 & & & & -2 & -1 & -2 & & & & & & & & & \\
& 1 & & & & -2 & -1 & -2 & & & & & & & & \\
& & 1 & & -2 & & -2 & -1 & & & & & & & & \\
& & & 1 & -1 & -2 & & -2 & & & & & & & & \\
& & & & 1 & & & & -2 & -1 & -2 & & & & & \\
& & & & & 1 & & & & -2 & -1 & -2 & & & & \\
& & & & & & 1 & & -2 & & -2 & -1 & & & & \\
& & & & & & & 1 & -1 & -2 & & -2 & & & & \\
& & & & & & & & 1 & & & & -2 & -1 & -2 & \\
& & & & & & & & & 1 & & & & -2 & -1 & -2 \\
& & & & & & & & & & 1 & & -2 & & -2 & -1 \\
& & & & & & & & & & & 1 & -1 & -2 & & -2 \\
-2 & -1 & -2 & & & & & & & & & & 1 & & & \\
& -2 & -1 & -2 & & & & & & & & & & 1 & & \\
-2 & & -2 & -1 & & & & & & & & & & & 1 & \\
-1 & -2 & & -2 & & & & & & & & & & & & 1
\end{bmatrix}$$

$$\begin{bmatrix}
1 & & & & & & & & & & & & & & & \\
& 1 & & & & -2 & -1 & -2 & & & & & & & & \\
& & 1 & & -2 & & -2 & -1 & & & & & & & & \\
& & & 1 & -1 & -2 & & -2 & & & & & & & & \\
& & & & 1 & & & & -2 & -1 & -2 & & & & & \\
& & & & & 1 & & & & -2 & -1 & -2 & & & & \\
& & & & & & 1 & & -2 & & -2 & -1 & & & & \\
& & & & & & & 1 & -1 & -2 & & -2 & & & & \\
& & & & & & & & 1 & & & & -2 & -1 & -2 & \\
& & & & & & & & & 1 & & & & -2 & -1 & -2 \\
& & & & & & & & & & 1 & & -2 & & -2 & -1 \\
& & & & & & & & & & & 1 & -1 & -2 & & -2 \\
& -1 & -2 & & 1 & -2 & 1 & & & & & & 1 & & & \\
& -2 & -1 & -2 & & & & & & & & & & 1 & & \\
& & -2 & -1 & 1 & -2 & 1 & & & & & & & & 1 & \\
& -2 & & -2 & -2 & -1 & -2 & & & & & & & & & 1
\end{bmatrix}$$

$$\begin{bmatrix}
1 & & & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & & & \\
 & & 1 & & -2 & & -2 & -1 & & & & & & & & \\
 & & & 1 & -1 & -2 & & -2 & & & & & & & & \\
 & & & & 1 & & & & -2 & -1 & -2 & & & & & \\
 & & & & & 1 & & & & -2 & -1 & -2 & & & & \\
 & & & & & & 1 & & -2 & & -2 & -1 & & & & \\
 & & & & & & & 1 & -1 & -2 & & -2 & & & & \\
 & & & & & & & & 1 & & & & -2 & -1 & -2 & \\
 & & & & & & & & & 1 & & & & -2 & -1 & -2 \\
 & & & & & & & & & & 1 & & -2 & & -2 & -1 \\
 & & & & & & & & & & & 1 & -1 & -2 & & -2 \\
 & & -2 & & 1 & 1 & & -2 & & & & & 1 & & & \\
 & & -1 & -2 & & 1 & -2 & 1 & & & & & & 1 & & \\
 & & -2 & -1 & 1 & -2 & 1 & & & & & & & & 1 & \\
 & & & -2 & -2 & & 1 & 1 & & & & & & & & 1
\end{bmatrix}$$

$$\begin{bmatrix}
1 & & & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & & & \\
 & & & 1 & -1 & -2 & & -2 & & & & & & & & \\
 & & & & 1 & & & & -2 & -1 & -2 & & & & & \\
 & & & & & 1 & & & & -2 & -1 & -2 & & & & \\
 & & & & & & 1 & & -2 & & -2 & -1 & & & & \\
 & & & & & & & 1 & -1 & -2 & & -2 & & & & \\
 & & & & & & & & 1 & & & & -2 & -1 & -2 & \\
 & & & & & & & & & 1 & & & & -2 & -1 & -2 \\
 & & & & & & & & & & 1 & & -2 & & -2 & -1 \\
 & & & & & & & & & & & 1 & -1 & -2 & & -2 \\
 & & & & 2 & 1 & 1 & 1 & & & & & 1 & & & \\
 & & & -2 & -2 & 1 & 1 & & & & & & & 1 & & \\
 & & & -1 & 2 & -2 & 2 & -2 & & & & & & & 1 & \\
 & & & -2 & -2 & & 1 & 1 & & & & & & & & 1
\end{bmatrix}$$

$$
\begin{bmatrix}
1 & & & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & & & \\
 & & & & 1 & & & & -2 & -1 & -2 & & & & & \\
 & & & & & 1 & & & & -2 & -1 & -2 & & & & \\
 & & & & & & 1 & & -2 & & -2 & -1 & & & & \\
 & & & & & & & 1 & -1 & -2 & & -2 & & & & \\
 & & & & & & & & 1 & & & & -2 & -1 & -2 & \\
 & & & & & & & & & 1 & & & & -2 & -1 & -2 \\
 & & & & & & & & & & 1 & & -2 & & -2 & -1 \\
 & & & & & & & & & & & 1 & -1 & -2 & & -2 \\
 & & & & 2 & 1 & 1 & 1 & & & & & 1 & & & \\
 & & & & 1 & 2 & 1 & 1 & & & & & & 1 & & \\
 & & & & 1 & 1 & 2 & 1 & & & & & & & 1 & \\
 & & & & 1 & 1 & 1 & 2 & & & & & & & & 1
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & & & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & & & \\
 & & & & 1 & & & & & & & & & & & \\
 & & & & & 1 & & & & -2 & -1 & -2 & & & & \\
 & & & & & & 1 & & -2 & & -2 & -1 & & & & \\
 & & & & & & & 1 & -1 & -2 & & -2 & & & & \\
 & & & & & & & & 1 & & & & -2 & -1 & -2 & \\
 & & & & & & & & & 1 & & & & -2 & -1 & -2 \\
 & & & & & & & & & & 1 & & -2 & & -2 & -1 \\
 & & & & & & & & & & & 1 & -1 & -2 & & -2 \\
 & & & & & 1 & 1 & 1 & -1 & 2 & -1 & & 1 & & & \\
 & & & & & 2 & 1 & 1 & 2 & 1 & 2 & & & 1 & & \\
 & & & & & 1 & 2 & 1 & 2 & 1 & 2 & & & & 1 & \\
 & & & & & 1 & 1 & 2 & 2 & 1 & 2 & & & & & 1
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 &&&&&&&&&&&&&&& \\
& 1 &&&&&&&&&&&&&& \\
&& 1 &&&&&&&&&&&&& \\
&&& 1 &&&&&&&&&&&& \\
&&&& 1 &&&&&&&&&&& \\
&&&&& 1 &&&&&&&&&& \\
&&&&&& 1 && -2 && -2 & -1 &&&& \\
&&&&&&& 1 & -1 & -2 && -2 &&&& \\
&&&&&&&& 1 &&&& -2 & -1 & -2 & \\
&&&&&&&&& 1 &&&& -2 & -1 & -2 \\
&&&&&&&&&& 1 && -2 && -2 & -1 \\
&&&&&&&&&&& 1 & -1 & -2 && -2 \\
&&&&&& 1 & 1 & -1 & -1 && 2 & 1 &&& \\
&&&&&& 1 & 1 & 2 && -1 & -1 && 1 && \\
&&&&&& 2 & 1 & 2 & -2 & -2 & 2 &&& 1 & \\
&&&&&& 1 & 2 & 2 & -2 & -2 & 2 &&&& 1
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 &&&&&&&&&&&&&&& \\
& 1 &&&&&&&&&&&&&& \\
&& 1 &&&&&&&&&&&&& \\
&&& 1 &&&&&&&&&&&& \\
&&&& 1 &&&&&&&&&&& \\
&&&&& 1 &&&&&&&&&& \\
&&&&&& 1 &&&&&&&&& \\
&&&&&&& 1 & -1 & -2 && -2 &&&& \\
&&&&&&&& 1 &&&& -2 & -1 & -2 & \\
&&&&&&&&& 1 &&&& -2 & -1 & -2 \\
&&&&&&&&&& 1 && -2 && -2 & -1 \\
&&&&&&&&&&& 1 & -1 & -2 && -2 \\
&&&&&&& 1 & 1 & -1 & 2 & -2 & 1 &&& \\
&&&&&&& 1 & -1 && 1 &&& 1 && \\
&&&&&&& 1 & 1 & -2 & 2 & -1 &&& 1 & \\
&&&&&&& 2 & -1 & -2 && -2 &&&& 1
\end{bmatrix}
$$

$$\begin{bmatrix}
1 &&&&&&&&&&&&&&& \\
& 1 &&&&&&&&&&&&&& \\
&& 1 &&&&&&&&&&&&& \\
&&& 1 &&&&&&&&&&&& \\
&&&& 1 &&&&&&&&&&& \\
&&&&& 1 &&&&&&&&&& \\
&&&&&& 1 &&&&&&&&& \\
&&&&&&& 1 &&&&&&&& \\
&&&&&&&& 1 &&&& -2 & -1 & -2 & \\
&&&&&&&&& 1 &&&& -2 & -1 & -2 \\
&&&&&&&&&& 1 && -2 && -2 & -1 \\
&&&&&&&&&&& 1 & -1 & -2 && -2 \\
&&&&&&&& 2 & 1 & 2 && 1 &&& \\
&&&&&&&&& 2 & 1 & 2 && 1 && \\
&&&&&&&& 2 && 2 & 1 &&& 1 & \\
&&&&&&&& 1 & 2 && 2 &&&& 1
\end{bmatrix}$$

$$\begin{bmatrix}
1 &&&&&&&&&&&&&&& \\
& 1 &&&&&&&&&&&&&& \\
&& 1 &&&&&&&&&&&&& \\
&&& 1 &&&&&&&&&&&& \\
&&&& 1 &&&&&&&&&&& \\
&&&&& 1 &&&&&&&&&& \\
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&&&&&&&& 1 &&&&&&& \\
&&&&&&&&& 1 &&&& -2 & -1 & -2 \\
&&&&&&&&&& 1 && -2 && -2 & -1 \\
&&&&&&&&&&& 1 & -1 & -2 && -2 \\
&&&&&&&&& 1 & 2 &&& 2 & -1 & \\
&&&&&&&&& 2 & 1 & 2 && 1 && \\
&&&&&&&&&& 2 & 1 & -1 & 2 && \\
&&&&&&&&& 2 && 2 & 2 & 1 & 2 & 1
\end{bmatrix}$$

$$\begin{bmatrix}
1 &&&&&&&&&&&&&&& \\
& 1 &&&&&&&&&&&&&& \\
&& 1 &&&&&&&&&&&&& \\
&&& 1 &&&&&&&&&&&& \\
&&&& 1 &&&&&&&&&&& \\
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&&&&&&&& 1 &&&&&&& \\
&&&&&&&&& 1 &&&&&& \\
&&&&&&&&&& 1 & & -2 & & -2 & -1 \\
&&&&&&&&&& & 1 & -1 & -2 & & -2 \\
&&&&&&&&&& 2 & & & -1 & & 2 \\
&&&&&&&&&& 1 & 2 & & & 2 & -1 \\
&&&&&&&&&& 2 & 1 & -1 & 2 & & \\
&&&&&&&&&& & 2 & 2 & & -1 &
\end{bmatrix}$$

$$\begin{bmatrix}
1 &&&&&&&&&&&&&&& \\
& 1 &&&&&&&&&&&&&& \\
&& 1 &&&&&&&&&&&&& \\
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&&&& 1 &&&&&&&&&&& \\
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&&&&&&&&&& 1 &&&&& \\
&&&&&&&&&&& 1 & -1 & -2 & & -2 \\
&&&&&&&&&&& & -1 & -1 & -1 & -1 \\
&&&&&&&&&&& 2 & 2 & & -1 & \\
&&&&&&&&&&& 1 & -2 & 2 & -1 & 2 \\
&&&&&&&&&&& 2 & 2 & & -1 &
\end{bmatrix}$$

$$
\begin{bmatrix}
1 &&&&&&&&&&&&&&& \\
& 1 &&&&&&&&&&&&&& \\
&& 1 &&&&&&&&&&&&& \\
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&&&&&&&&&&& 1 &&&& \\
&&&&&&&&&&&& -1 & -1 & -1 & -1 \\
&&&&&&&&&&&& -1 & -1 & -1 & -1 \\
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&&&&&&&&&&&& -1 & -1 & -1 & -1
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 &&&&&&&&&&&&&&& \\
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&&&&&&&&&&&& 1 &&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&&
\end{bmatrix}
$$

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